

BSC Ag First Year First Sem

Introductory Mathematics

Arithmetic Progression (AP):

An **arithmetic progression (AP)** is a sequence of numbers in which the difference between any two consecutive terms is constant. This difference is called the **common difference (d)**.

General form of an AP: $a, a + d, a + 2d, a + 3d, \dots$

Where: a is the first term, d is the common difference.

Nth Term of an AP:

The n th term of an AP, denoted T_n , is given by: $T_n = a + (n - 1)d$

Where:

a is the first term, d is the common difference, n is the number of terms.

Sum of First N Terms of an AP:

The sum of the first n terms, denoted S_n , is given by:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Or, equivalently: $S_n = \frac{n}{2}(a + l)$ Where l is the last term.

1. Arithmetic Progression (AP)

Given: First term $a = 3$
Common difference $d = 5$
Number of terms $n = 5$

Sequence: Using the formula for the n th term: $T_n = a + (n - 1)d$

The first five terms of the AP are:

$$T_1 = 3 + (1 - 1) \cdot 5 = 3$$

$$T_2 = 3 + (2 - 1) \cdot 5 = 3 + 5 = 8$$

$$T_3 = 3 + (3 - 1) \cdot 5 = 3 + 10 = 13$$

$$T_4 = 3 + (4 - 1) \cdot 5 = 3 + 15 = 18$$

$$T_5 = 3 + (5 - 1) \cdot 5 = 3 + 20 = 23$$

The AP is: 3, 8, 13, 18, 23.

If the first term $a = 5$ and the common difference $d = 3$, then the first few terms of the arithmetic progression are:

$$5, 8, 11, 14, 17, \dots$$

The 5th term can be calculated as:

$$T_5 = 5 + (5 - 1) \cdot 3 = 5 + 12 = 17$$

The sum of the first 5 terms can be calculated as:

$$S_5 = \frac{5}{2} \cdot (2 \cdot 5 + (5 - 1) \cdot 3) = \frac{5}{2} \cdot (10 + 12) = \frac{5}{2} \cdot 22 = 55$$

Question 1:

The first term of an arithmetic progression is 7, and the common difference is 5. Find the 10th term of the sequence.

Solution: We know the formula for the n -th term of an AP:

$$T_n = a + (n - 1) \cdot d$$

Given: $a = 7$ (first term), $d = 5$ (common difference), $n = 10$

(we need to find the 10th term).

Substitute the values into the formula:

$$T_{10} = 7 + (10 - 1) \cdot 5 = 7 + 9 \cdot 5 = 7 + 45 = 52$$

So, the 10th term is 52.

Find the sum of the first 20 terms of an arithmetic progression whose first term is 3 and common difference is 4.

Solution:

We know the sum of the first n terms of an AP is given by:

$$S_n = \frac{n}{2} \cdot (2a + (n - 1) \cdot d)$$

Given: $a = 3$ (first term), $d = 4$ (common difference), $n = 20$ (number of terms).

Substitute the values:

$$S_{20} = \frac{20}{2} \cdot (2 \cdot 3 + (20 - 1) \cdot 4) = 10 \cdot (6 + 19 \cdot 4) = 10 \cdot (6 + 76) = 10 \cdot 82 = 820$$

So, the sum of the first 20 terms is **820**.

Question 3:

The 5th term of an arithmetic progression is 15, and the 12th term is 50. Find the common difference and the first term.

Solution:

We use the formula for the n -th term of an AP:

$$T_n = a + (n - 1) \cdot d$$

For the 5th term: $T_5 = a + 4 \cdot d = 15 \quad (1)$

For the 12th term: $T_{12} = a + 11 \cdot d = 50 \quad (2)$

Now, subtract equation (1) from equation (2):

$$(a + 11d) - (a + 4d) = 50 - 15$$

$$7d = 35 \quad \Rightarrow \quad d = \frac{35}{7} = 5$$

Now, substitute $d = 5$ back into equation (1):

$$a + 4 \cdot 5 = 15 \quad \Rightarrow \quad a + 20 = 15 \quad \Rightarrow \quad a = 15 - 20 = -5$$

So, the first term is -5, and the common difference is 5.

Question 4:

How many terms are there in the arithmetic progression 6, 9, 12, ..., 60?

Solution:

We use the formula for the n -th term:

$$T_n = a + (n - 1) \cdot d$$

Given: $a = 6$ (first term), $d = 3$ (common difference), $T_n = 60$ (last term).

Substitute into the formula: $60 = 6 + (n - 1) \cdot 3$

$$60 - 6 = (n - 1) \cdot 3 \quad \Rightarrow \quad 54 = (n - 1) \cdot 3$$

$$n - 1 = \frac{54}{3} = 18 \quad \Rightarrow \quad n = 18 + 1 = 19$$

So, there are **19 terms** in the sequence.

Question 5:

If the sum of the first 15 terms of an arithmetic progression is 120, and the first term is 3, find the common difference.

Solution: We use the sum formula:

$$S_n = \frac{n}{2} \cdot (2a + (n - 1) \cdot d)$$

Given:

$$S_{15} = 120, \quad a = 3, \quad n = 15.$$

Substitute into the formula:

$$120 = \frac{15}{2} \cdot (2 \cdot 3 + (15 - 1) \cdot d)$$

$$120 = \frac{15}{2} \cdot (6 + 14d)$$

$$240 = 15 \cdot (6 + 14d) \quad \Rightarrow \quad 240 = 90 + 210d$$

$$240 - 90 = 210d \quad \Rightarrow \quad 150 = 210d \quad \Rightarrow \quad d = \frac{150}{210} = \frac{5}{7}$$

So, the common difference is $\frac{5}{7}$.

PRACTICE QUESTIONS

Question 6:

The 8th term of an arithmetic progression is 20, and the 15th term is 41. Find the first term and the common difference.

Question 7:

Find the sum of the first 50 terms of the arithmetic progression: 12, 10, 8, ...

Question 8:

If the sum of the first 25 terms of an arithmetic progression is 525 and the common difference is 2, find the first term.

Question 9:

The first term of an arithmetic progression is 2, and the last term is 29. If the sum of all the terms is 155, find the number of terms in the arithmetic progression.

Question 10:

In an arithmetic progression, the first term is 7, and the 6th term is 22. How many terms are there if the last term is 97?

Question 11:

If the 3rd and the 7th terms of an arithmetic progression are 10 and 26, respectively, find the common difference and the first term of the progression.

Question 12:

The sum of the first 16 terms of an arithmetic progression is 256, and the common difference is 2. Find the first term.

Question 13:

In an arithmetic progression, the 9th term is 17, and the 14th term is 32. Find the sum of the first 20 terms.

Question 14:

Find three numbers in an arithmetic progression whose sum is 33 and whose product is 792.

Question 16:

The 5th term of an arithmetic progression is 25, and the sum of the first 10 terms is 230. Find the first term and the common difference.

Question 17:

In an arithmetic progression, the 4th term is 18, and the 12th term is 50. Find the sum of the first 15 terms.

Question 18:

The first term of an arithmetic progression is 5, and the sum of the first 20 terms is 610. Find the common difference.

Question 19:

The sum of the first 20 terms of an arithmetic progression is 400, and the first term is 7. Find the common difference.

Geometric Progression

A **geometric progression** (GP) is a sequence of numbers where each term after the first is obtained by multiplying the previous term by a fixed, non-zero number called the **common ratio** (r). The general form of a geometric progression can be written as:

$$a, ar, ar^2, ar^3, ar^4, \dots$$

Where: a is the first term, r is the common ratio, and

The n -th term of the progression is given by $a \cdot r^{(n-1)}$.

Example:

If $a = 3$ and $r = 2$, the sequence will be: 3, 6, 12, 24, 48, ...

Sum of the first n terms:
$$S_n = a \frac{1 - r^n}{1 - r} \quad (\text{if } r \neq 1)$$

Sum of an infinite geometric series (if $|r| < 1$):
$$S_{\infty} = \frac{a}{1 - r}$$

Example Problem:

Consider a geometric progression where the first term $a = 4$ and the common ratio $r = 3$.

- a) Find the first 5 terms of the progression. b) Find the 6th term of the progression.
c) Find the sum of the first 6 terms.

Solution:

a) First 5 terms of the progression:

The terms of a geometric progression are calculated using the formula:

$$T_n = a \cdot r^{(n-1)}$$

For the given GP: $a = 4$, $r = 3$.

Now, calculate the first 5 terms:

$$T_1 = 4 \cdot 3^{(1-1)} = 4 \cdot 3^0 = 4 \cdot 1 = 4$$

$$T_2 = 4 \cdot 3^{(2-1)} = 4 \cdot 3^1 = 4 \cdot 3 = 12$$

$$T_3 = 4 \cdot 3^{(3-1)} = 4 \cdot 3^2 = 4 \cdot 9 = 36$$

$$T_4 = 4 \cdot 3^{(4-1)} = 4 \cdot 3^3 = 4 \cdot 27 = 108$$

$$T_5 = 4 \cdot 3^{(5-1)} = 4 \cdot 3^4 = 4 \cdot 81 = 324$$

Thus, the first 5 terms of the GP are: **4, 12, 36, 108, 324.**

b) Finding the 6th term:

Use the formula for the n -th term: $T_n = a \cdot r^{(n-1)}$

For $n = 6$:

$$T_6 = 4 \cdot 3^{(6-1)} = 4 \cdot 3^5 = 4 \cdot 243 = 972$$

So, the 6th term is **972**.

c) Sum of the first 6 terms:

The sum of the first n terms of a geometric progression is given by the formula:

$$S_n = a \cdot \frac{1 - r^n}{1 - r} \quad (\text{for } r \neq 1)$$

For the first 6 terms:

$$S_6 = 4 \cdot \frac{1 - 3^6}{1 - 3} = 4 \cdot \frac{1 - 729}{1 - 3} = 4 \cdot \frac{-728}{-2} = 4 \cdot 364 = 1456$$

So, the sum of the first 6 terms is **1456**.

4. Infinite Geometric Series

If the common ratio r is between -1 and 1 (i.e., $|r| < 1$), the sum of an **infinite geometric series** can be calculated. For example, if $a = 10$ and $r = 0.5$:

$$S_{\infty} = \frac{a}{1 - r} = \frac{10}{1 - 0.5} = \frac{10}{0.5} = 20$$

The sum of the infinite series is **20**.

1. Find the 5th term of the geometric progression with first term 3 and common ratio 2.
2. What is the 6th term of the geometric progression where the first term is 5 and the common ratio is $\frac{1}{2}$?
3. If the 3rd term of a GP is 24 and the common ratio is 2, what is the first term?
4. Find the sum of the first 4 terms of a GP where the first term is 2 and the common ratio is 3.
5. If the sum of the first 3 terms of a GP is 13 and the first term is 3, find the common ratio.
6. The sum of the first 5 terms of a GP is 31, and the common ratio is 2. Find the first term.
7. Calculate the 7th term of the GP: 5, 10, 20, ...
8. Find the common ratio of the GP where the 4th term is 16 and the first term is 2.
9. What is the sum of the infinite GP where the first term is 4 and the common ratio is $\frac{1}{3}$?
10. A GP has a first term of 8 and a common ratio of 0.5. Find the 5th term.
11. The first term of a GP is 6, and the common ratio is 2. Find the sum of the first 5 terms.
12. If the sum of an infinite GP is 15 and the common ratio is $\frac{2}{3}$, what is the first term?
13. The sum of the first 6 terms of a GP is 126, and the common ratio is 2. Find the first term.
14. Find the common ratio of the GP where the 2nd term is 4 and the 5th term is 32.
15. The 4th term of a GP is 54, and the common ratio is 3. Find the first term.
16. Calculate the sum of the first 7 terms of a GP where $a = 1$ and $r = 0.5$.
17. What is the sum of the first 3 terms of the GP: 2, 4, 8, ...?
18. The sum of the first 4 terms of a GP is 120, and the common ratio is 2. Find the first term.
19. Find the common ratio of a GP if the sum of the first 3 terms is 21 and the first term is 7.
20. If the sum of an infinite GP is 18 and the first term is 6, find the common ratio.
21. The sum of the first 6 terms of a GP is 189, and the common ratio is 3. Find the first term.
22. The 8th term of a GP is 512, and the common ratio is 2. Find the first term.
23. What is the 4th term of the GP: 7, 21, 63, ...?

24. The first term of a GP is 12, and the sum of the first 3 terms is 39. Find the common ratio.
25. Find the sum of the first 5 terms of a GP where $a = 4$ and $r = 2$.
26. The sum of an infinite GP is 20, and the common ratio is 0.8. Find the first term.
27. The 5th term of a GP is 243, and the first term is 3. Find the common ratio.
28. Find the common ratio of a GP where the 3rd term is 16 and the 6th term is 128.
29. The sum of the first 4 terms of a GP is 60, and the common ratio is 4. Find the first term.
30. If the sum of the first 5 terms of a GP is 62, and the common ratio is 2, find the first term.

Answers:

1. $T_5 = 3 \cdot 2^{5-1} = 48$
2. $T_6 = 5 \cdot \left(\frac{1}{2}\right)^{6-1} = 5 \cdot \frac{1}{32} = \frac{5}{32}$
3. $T_3 = a \cdot r^2 = 24 \implies a \cdot 2^2 = 24 \implies a = 6$
4. $S_4 = 2 \cdot \frac{1-3^4}{1-3} = 2 \cdot \frac{1-81}{-2} = 82$
5. $S_3 = 3 \cdot \frac{1-r^3}{1-r} = 13 \implies r = 1/2$
6. $S_5 = a \cdot \frac{1-2^5}{1-2} = 31 \implies a = 1$
7. $T_7 = 5 \cdot 2^{7-1} = 5 \cdot 64 = 320$
8. $T_4 = a \cdot r^3 = 16 \implies 2 \cdot r^3 = 16 \implies r = 2$
9. $S_\infty = \frac{4}{1-\frac{1}{3}} = 6$
10. $T_5 = 8 \cdot 0.5^{5-1} = 8 \cdot 0.0625 = 0.5$
11. $S_5 = 6 \cdot \frac{1-2^5}{1-2} = 6 \cdot 31 = 186$
12. $S_\infty = \frac{a}{1-\frac{2}{3}} = 15 \implies a = 5$
13. $S_6 = a \cdot \frac{1-2^6}{1-2} = 126 \implies a = 2$
14. $T_5 = T_2 \cdot r^3 \implies 32 = 4 \cdot r^3 \implies r = 2$
15. $T_4 = a \cdot r^3 = 54 \implies a \cdot 3^3 = 54 \implies a = 2$
16. $S_7 = 1 \cdot \frac{1-0.5^7}{1-0.5} \approx 1.992$
17. $S_3 = 2 \cdot \frac{1-2^3}{1-2} = 14$
18. $S_4 = a \cdot \frac{1-2^4}{1-2} = 120 \implies a = 8$
19. $S_3 = 7 \cdot \frac{1-r^3}{1-r} = 21 \implies r = 1/2$
20. $S_\infty = \frac{6}{1-r} = 18 \implies r = 2/3$
21. $S_6 = a \cdot \frac{1-3^6}{1-3} = 189 \implies a = 1$
22. $T_8 = a \cdot 2^7 = 512 \implies a = 4$
23. $T_4 = 7 \cdot 3^{4-1} = 189$
25. $S_5 = 4 \cdot \frac{1-2^5}{1-2} = 124$
26. $S_\infty = \frac{a}{1-0.8} = 20 \implies a = 4$
27. $T_5 = 3 \cdot r^{5-1} = 243 \implies r = 3$
28. $T_6 = T_3 \cdot r^3 \implies 128 = 16 \cdot r^3 \implies r = 2$
29. $S_4 = a \cdot \frac{1-4^4}{1-4} = 60 \implies a = 4$
30. $S_5 = a \cdot \frac{1-2^5}{1-2} = 62 \implies a = 2$

Q- 24

The sum of the first 3 terms in a GP can be written as:

$$S_3 = a + ar + ar^2$$

Given:

$$S_3 = 39$$

Substitute $a = 12$:

$$12 + 12r + 12r^2 = 39$$

Now, divide the entire equation by 12 to simplify:

$$1 + r + r^2 = \frac{39}{12}$$

Simplify $\frac{39}{12}$ to get:

$$1 + r + r^2 = 3.25$$

This gives the quadratic equation:

$$r^2 + r + 1 - 3.25 = 0$$

$$r^2 + r + 1 - 3.25 = 0$$

$$r^2 + r - 2.25 = 0$$

Now, solve this quadratic equation for r using the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 1$, and $c = -2.25$.

Substitute the values:

$$r = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-2.25)}}{2 \cdot 1}$$

$$r = \frac{-1 \pm \sqrt{1 + 9}}{2}$$

$$r = \frac{-1 \pm \sqrt{10}}{2}$$

So the possible values for r are:

$$r = \frac{-1 + \sqrt{10}}{2} \quad \text{or} \quad r = \frac{-1 - \sqrt{10}}{2}$$

Since the common ratio is typically positive in practical contexts, we take:

$$r = \frac{-1 + \sqrt{10}}{2} \approx 0.58$$

Thus, the common ratio r is approximately 0.58.

Harmonic Progression (HP):

A harmonic progression (HP) is a sequence of numbers where the reciprocals of the terms form an arithmetic progression.

General form of an HP:

If the sequence $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots$ forms an AP, then the original sequence forms an HP.

Nth Term of an HP:

If the terms of an HP are reciprocals of an AP, the n th term T_n of the HP is given by the reciprocal of the n th term of the corresponding AP:

$$T_n = \frac{1}{a + (n - 1)d}$$

Where a and d come from the corresponding AP.

The **sum of a Harmonic Progression (HP)** doesn't have a straightforward formula like Arithmetic Progression (AP) or Geometric Progression (GP), as it involves reciprocals of terms. However, to compute the sum of an HP, you generally sum the reciprocals of an **Arithmetic Progression (AP)** since the terms of an HP are the reciprocals of an AP.

Steps to find the sum of an HP:

Given an HP sequence $a_1, a_2, a_3, \dots, a_n$, you can sum the reciprocals of its terms, which will form an AP.

Let the corresponding AP be: $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots, \frac{1}{a_n}$

This AP has the first term $A_1 = \frac{1}{a_1}$ and the common difference d .

Formula for the sum of reciprocals (AP):

The sum of the first n terms of an AP is given by:

$$S_n = \frac{n}{2} (2A_1 + (n - 1)d)$$

Where: S_n is the sum of the first n terms of the HP,

$A_1 = \frac{1}{a_1}$ is the first term of the AP formed by the reciprocals,

d is the common difference of the AP formed by the reciprocals.

3. Harmonic Progression (HP)

A harmonic progression is derived from an arithmetic progression by taking the reciprocals of the terms.

Given: The AP: 2, 6, 10, 14, 18

The reciprocals of these terms form an HP: $\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \frac{1}{18}$

Nth term of the HP:

The nth term of the HP is the reciprocal of the nth term of the corresponding AP: For $n = 3$, the 3rd term of the AP is 10, so the 3rd term of the HP is:

$$T_3 = \frac{1}{10}$$

Sum of the first 5 terms of the HP: $S_5 = \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \frac{1}{18}$

Finding the sum: $S_5 = 0.5 + 0.1667 + 0.1 + 0.0714 + 0.0556 = 0.8937$

Question 1:

Find the 5th term of a harmonic progression whose first term is 2 and the common difference of the corresponding arithmetic progression is 0.5.

Solution:

The terms of an HP are reciprocals of an AP.

Let the corresponding AP have first term $A_1 = \frac{1}{2}$ (since the first term of HP is 2) and common difference $d = 0.5$.

The n th term of the AP is given by:

$$A_n = A_1 + (n - 1)d = \frac{1}{2} + (n - 1) \times 0.5$$

For $n = 5$:

$$A_5 = \frac{1}{2} + (5 - 1) \times 0.5 = \frac{1}{2} + 2 = 2.5$$

The 5th term of the HP is the reciprocal of A_5 :

$$T_5 = \frac{1}{2.5} = 0.4$$

Thus, the 5th term of the HP is **0.4**.

Question 2:

Find the sum of the first 4 terms of a harmonic progression if the first term is 3 and the common difference of the corresponding arithmetic progression is 1.

Solution:

Let the first term of the corresponding AP be $A_1 = \frac{1}{3}$ and the common difference $d = 1$.

The terms of the corresponding AP are:

$$A_1 = \frac{1}{3}, \quad A_2 = A_1 + d = \frac{1}{3} + 1 = \frac{4}{3}, \quad A_3 = A_2 + d = \frac{7}{3}, \quad A_4 = A_3 + d = \frac{10}{3}$$

The terms of the HP are the reciprocals of these values:

$$T_1 = 3, \quad T_2 = \frac{3}{4}, \quad T_3 = \frac{3}{7}, \quad T_4 = \frac{3}{10}$$

Now, sum the first 4 terms:

$$S_4 = 3 + \frac{3}{4} + \frac{3}{7} + \frac{3}{10}$$

We can compute this sum:

$$S_4 = 3 + 0.75 + 0.4286 + 0.3 = 4.4786$$

Thus, the sum of the first 4 terms of the HP is approximately **4.48**.

Question 3:

The first three terms of a harmonic progression are a, b, c . If the first term is 2 and the sum of the first three terms is 6, find the values of b and c .

Solution:

1. Let the terms of the HP be:

$$a_1 = a = 2$$

Let the second term be b

Let the third term be c

2. Using the sum: $a + b + c = 6$

Substituting $a = 2$:

$$2 + b + c = 6 \implies b + c = 4$$

3. Reciprocals form an AP: $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

Using the property of AP: $2 \cdot \frac{1}{b} = \frac{1}{a} + \frac{1}{c}$

Substitute $a = 2$: $2 \cdot \frac{1}{b} = \frac{1}{2} + \frac{1}{c}$

Multiply through by $2bc$: $4c = bc + 2b$

4. Solve Equations (1) and (2): From equation (1), express c :

$$c = 4 - b$$

Substitute c in equation (2): $4(4 - b) = b(4 - b) + 2b$

$$16 - 4b = 4b - b^2 + 2b$$

$$16 - 4b = -b^2 + 6b$$

Rearranging gives: $b^2 - 10b + 16 = 0$

5. Solve the quadratic: Using the quadratic formula $b = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$:

$$b = \frac{10 \pm \sqrt{(-10)^2 - 4 \cdot 1 \cdot 16}}{2 \cdot 1} = \frac{10 \pm \sqrt{100 - 64}}{2} = \frac{10 \pm \sqrt{36}}{2}$$

$$b = \frac{10 \pm 6}{2} \implies b = 8 \text{ or } 2$$

6. Finding c : If $b = 8$:

$$c = 4 - 8 = -4 \text{ (not valid)}$$

If $b = 2$:

If $b = 2$:

$$c = 4 - 2 = 2$$

Answer: The values of b and c are 2 (both) as the terms in the HP are all equal.

Question 5:

Find the sum of the first 4 terms of a harmonic progression where the first term is 5 and the common difference of the corresponding arithmetic progression is 1.

Solution:

1. First term of the HP $a_1 = 5$.
2. First term of the corresponding AP $A_1 = \frac{1}{5}$.
3. Common difference $d = 1$.

Now, calculate the terms of the corresponding AP:

$$A_2 = A_1 + d = \frac{1}{5} + 1 = \frac{1}{5} + \frac{5}{5} = \frac{6}{5},$$

$$A_3 = A_1 + 2d = \frac{1}{5} + 2 = \frac{1}{5} + \frac{10}{5} = \frac{11}{5},$$

$$A_4 = A_1 + 3d = \frac{1}{5} + 3 = \frac{1}{5} + \frac{15}{5} = \frac{16}{5}.$$

Now, find the first 4 terms of the HP by taking the reciprocals of the AP terms:

$$T_1 = 5, \quad T_2 = \frac{5}{6}, \quad T_3 = \frac{5}{11}, \quad T_4 = \frac{5}{16}.$$

Sum of the first 4 terms of the HP:

$$S = 5 + \frac{5}{6} + \frac{5}{11} + \frac{5}{16}.$$

Convert each fraction to a common denominator (which is 528):

$$S = 5 + \frac{440}{528} + \frac{240}{528} + \frac{165}{528} = 5 + \frac{845}{528}.$$

Simplify: $S = 5 + 1.6 \approx 6.6.$

Answer: The sum of the first 4 terms is approximately 6.6.

Question 9:

The first and second terms of a harmonic progression are 5 and 10, respectively. Find the third term of the progression.

Solution:

1. First term of the HP $a_1 = 5$, second term $a_2 = 10$.

2. The corresponding AP terms are:

$$A_1 = \frac{1}{5}, \quad A_2 = \frac{1}{10}.$$

3. The common difference d is:

$$d = A_2 - A_1 = \frac{1}{10} - \frac{1}{5} = \frac{1}{10} - \frac{2}{10} = -\frac{1}{10}.$$

Now, find the 3rd term of the AP:

Now, find the 3rd term of the AP:

$$A_3 = A_2 + d = \frac{1}{10} + \left(-\frac{1}{10}\right) = 0.$$

The 3rd term of the HP is the reciprocal of A_3 :

$$T_3 = \frac{1}{A_3} = \frac{1}{0}.$$

Since this is undefined, the third term of the HP **does not exist**.

MATRICES

An arrangement of numbers in rows and columns. A matrix of type “ $(m \times n)$ ” is defined as arrangement of $(m \times n)$ numbers in ‘ m ’ rows & ‘ n ’ columns. Usually these numbers are enclosed within square brackets [] (or) simple brackets () are denoted by capital letters A, B, C etc.

Example

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & -1 & 3 \\ 4 & 2 & 8 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 10 & 2 \\ 9 & -1 & 3 \\ 4 & 8 & 5 \end{pmatrix}$$

Here A is of type 3×4 & B is of type 4×3

Types of matrices

1. Row matrix: It is a matrix containing only one row and several columns .It is also called as row vector.

Example:

$$[1 \quad 3 \quad 7 \quad 9 \quad 6]$$



(1 x 5) matrix called row vector.

2. Column matrix: It is a matrix containing only one column. It is also known as column vector.

Example: $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} (3 \times 1)$

3. Square matrix: A matrix is called as square matrix, if the number of rows is equal to number of columns.

$$\begin{pmatrix} 4 & 2 & 4 \\ 1 & 9 & 8 \\ 6 & 5 & 2 \end{pmatrix}$$

Example

The elements a_{11} , a_{22} , a_{33} etc fall along the diagonal & this is called a leading diagonal (or) principal diagonal of the matrix.

4. Trace of the matrix

It is defined as the sum of the elements along the leading diagonal.

In this above matrix the trace of the matrix is

$$4 + 9 + 2 = 15.$$

5. Diagonal matrix

It is a square matrix in which all the elements other than in the leading diagonals are zero's.

Eg:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

6. Scalar matrix

It is a diagonal matrix in which all the elements in the leading diagonal are same.

Eg:
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

7. Unit matrix or identify matrix

It is a diagonal matrix, in which the elements along the leading diagonal are equal to one. It is denoted by I

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8. Zero matrix (or) Non-matrix

It is matrix all of whose elements are equal to zero denoted by "O"

$$\text{Eg: } O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad 2 \times 3 \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 2 \times 2$$

9. Triangular matrix

There are two types. 1. Lower Triangular Matrix 2. Upper Triangular Matrix.

Lower Triangular matrix

It is a square matrix in which all the elements above the leading diagonal are zeros.

Eg:
$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 6 & 2 \end{pmatrix}$$

Upper Triangular matrix

Square matrix in which all the elements below the leading diagonal are zeros

Eg:
$$\begin{pmatrix} 5 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

10. Symmetric matrix

A square matrix $A = \{a_{ij}\}$ said $i = 1$ to n ; $j = 1$ to n said to symmetric, if $a_{ij} = a_{ji}$ for all i and j .

Eg:
$$\begin{pmatrix} 1 & 3 & 4 \\ 3 & 6 & -5 \\ 4 & -5 & 2 \end{pmatrix}$$

11. Skew symmetric matrix

A square matrix $A = \{a_{ij}\}$ $i = 1$ to n is called skew symmetric, if $a_{ij} = -a_{ji}$ for all i & j . Here $a_{ii} = 0$ for all i

Eg:
$$\begin{pmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{pmatrix}$$

Algebra of matrices

1. Equality of matrices

Two matrices A & B are equal, if and only if,

- (i) Both A & B are of the same type
- (ii) Every element of 'B' is the same as the corresponding element of 'A'.

Example

1.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & -1 & 3 \\ 4 & 2 & 8 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 10 & 2 \\ 9 & -1 & 3 \\ 4 & 8 & 5 \end{pmatrix}$$

Here order of matrix A is not same as order matrix B, the two matrices are not equal.

$$A \neq B$$

2. Find the value of a and b given

$$\begin{bmatrix} 4 & 5 \\ a & b \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$$

Solution:

The given matrices are equal

$$\therefore a = 3, b = 2$$

2. Addition of matrices

Two matrices A & B can be added if and only if,

- (i) Both are of the same type.
- (ii) The resulting matrix of A & B is also of same type and is obtained by adding the all elements of 'A' to the corresponding elements of 'B'.

Example

1. Find $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4+2 & 5+3 \\ 5+2 & 6+1 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 7 & 7 \end{bmatrix}$$

3. Subtraction of the matrices

This can be done, when both the matrices are of same type.

(A-B) is obtained by subtracting the elements of 'A' with corresponding elements of 'B'.

Example

1. Find $\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$

Solution

$$\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4-2 & 5-3 \\ 5-2 & 6-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4-2 & 5-3 \\ 5-2 & 6-1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$$

4. Multiplication of matrix

They are of two types : 1. By a scalar $\rightarrow$ K B.

2. By a matrix $\rightarrow$ A x B.

i) Scalar multiplication

To multiply a matrix 'A' by a scalar 'K', then multiply every element of a matrix 'A' by that scalar.

Example

1. Find $2 \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}$

Solution:

$$2 \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 10 & 12 \end{bmatrix}$$

Matrix Multiplication

Matrix multiplication involves taking the dot product of the rows of the first matrix with the columns of the second matrix. It is essential that the **number of columns** of the first matrix is equal to the **number of rows** of the second matrix for the multiplication to be valid.

Example of Matrix Multiplication

Let's consider two matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad (2 \times 3 \text{ matrix})$$

$$B = \begin{pmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{pmatrix} \quad (3 \times 2 \text{ matrix})$$

Since matrix A has 3 columns and matrix B has 3 rows, matrix multiplication is possible. The result will be a 2×2 matrix.

To compute $C = A \times B$, we take the dot product of each row of A with each column of B .

1. First element (row 1 of A and column 1 of B):

$$C_{11} = (1 \times 7) + (2 \times 9) + (3 \times 11) = 7 + 18 + 33 = 58$$

2. Second element (row 1 of A and column 2 of B):

$$C_{12} = (1 \times 8) + (2 \times 10) + (3 \times 12) = 8 + 20 + 36 = 64$$

3. Third element (row 2 of A and column 1 of B):

$$C_{21} = (4 \times 7) + (5 \times 9) + (6 \times 11) = 28 + 45 + 66 = 139$$

4. Fourth element (row 2 of A and column 2 of B):

$$C_{22} = (4 \times 8) + (5 \times 10) + (6 \times 12) = 32 + 50 + 72 = 154$$

Thus, the product matrix C is:

$$C = \begin{pmatrix} 58 & 64 \\ 139 & 154 \end{pmatrix}$$

Suppose we have:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

To find $C = A \times B$:

1. $C[1, 1] = (1 \times 5) + (2 \times 7) = 5 + 14 = 19$
2. $C[1, 2] = (1 \times 6) + (2 \times 8) = 6 + 16 = 22$
3. $C[2, 1] = (3 \times 5) + (4 \times 7) = 15 + 28 = 43$
4. $C[2, 2] = (3 \times 6) + (4 \times 8) = 18 + 32 = 50$

So, the resulting matrix C is:

$$C = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

This process applies to any matrices, as long as the number of columns in the first matrix equals the number of rows in the second.

Transpose

The **transpose of a matrix** A is obtained by flipping it over its diagonal, meaning that the row and column indices of each element are swapped. If A is an $m \times n$ matrix (with m rows and n columns), its transpose, denoted by A^T , will be an $n \times m$ matrix.

For a matrix A :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The transpose A^T will be: $A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$

Properties of Transpose

1. Transpose of a Transpose: $(A^T)^T = A$.
2. Addition: $(A + B)^T = A^T + B^T$.
3. Multiplication: $(AB)^T = B^T A^T$.
4. Scalar Multiplication: $(cA)^T = cA^T$ where c is a scalar.
5. Symmetric Matrix: If $A = A^T$, then A is symmetric.

1. Find the transpose of the matrix: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
2. Given the matrix $B = \begin{bmatrix} 5 & 7 & 9 \\ 2 & 6 & 8 \end{bmatrix}$, compute B^T .
3. If $C = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 0 & -4 \end{bmatrix}$, find C^T .
4. Let $D = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$. Verify that $(D + E)^T = D^T + E^T$.
5. Given two matrices $F = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$ and $G = \begin{bmatrix} 0 & 2 \\ 5 & -3 \end{bmatrix}$, show that $(FG)^T = G^T F^T$.
6. For a scalar $k = 3$ and matrix $H = \begin{bmatrix} 2 & -5 \\ 4 & 1 \end{bmatrix}$, find $(kH)^T$ and verify that it equals kH^T .
8. Suppose $K = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $L = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$. Compute $(KL)^T$ and then verify it by finding $L^T K^T$.
9. Find the transpose of the product $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ by calculating both $(AB)^T$ and $B^T A^T$ to confirm that $(AB)^T = B^T A^T$.
10. Let $M = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$. Calculate $M \cdot M^T$ and $M^T \cdot M$. Are these products symmetric matrices?

Answers

1. Transpose of A : $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

2. Transpose of B : $B = \begin{bmatrix} 5 & 7 & 9 \\ 2 & 6 & 8 \end{bmatrix}, \quad B^T = \begin{bmatrix} 5 & 2 \\ 7 & 6 \\ 9 & 8 \end{bmatrix}$

3. Transpose of C : $C = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 0 & -4 \end{bmatrix}, \quad C^T = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 5 & -4 \end{bmatrix}$

4. Verify $(D + E)^T = D^T + E^T$:

$$D = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$D + E = \begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}$$

$$(D + E)^T = \begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}$$

$$D^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}, \quad E^T = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$D^T + E^T = \begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}$$

Both sides are equal, so $(D + E)^T = D^T + E^T$ is verified.

5. Show that $(FG)^T = G^T F^T$:

$$F = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 2 \\ 5 & -3 \end{bmatrix}$$

$$FG = \begin{bmatrix} (-5) & (-5) \\ 20 & -6 \end{bmatrix}, \quad (FG)^T = \begin{bmatrix} -5 & 20 \\ -5 & -6 \end{bmatrix}$$

$$G^T = \begin{bmatrix} 0 & 5 \\ 2 & -3 \end{bmatrix}, \quad F^T = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

$$G^T F^T = \begin{bmatrix} -5 & 20 \\ -5 & -6 \end{bmatrix}$$

Thus, $(FG)^T = G^T F^T$ is verified.

6. Verify $(kH)^T = kH^T$:

$$k = 3, \quad H = \begin{bmatrix} 2 & -5 \\ 4 & 1 \end{bmatrix}$$

$$kH = \begin{bmatrix} 6 & -15 \\ 12 & 3 \end{bmatrix}, \quad (kH)^T = \begin{bmatrix} 6 & 12 \\ -15 & 3 \end{bmatrix}$$

$$H^T = \begin{bmatrix} 2 & 4 \\ -5 & 1 \end{bmatrix}, \quad kH^T = \begin{bmatrix} 6 & 12 \\ -15 & 3 \end{bmatrix}$$

So, $(kH)^T = kH^T$ is verified.

8. Compute $(KL)^T$ and verify with $L^T K^T$:

$$K = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad L = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$KL = \begin{bmatrix} 3 & -4 & 5 \\ 5 & -8 & 13 \\ 7 & -12 & 21 \end{bmatrix}$$

$$(KL)^T = \begin{bmatrix} 3 & 5 & 7 \\ -4 & -8 & -12 \\ 5 & 13 & 21 \end{bmatrix}$$

$$L^T = \begin{bmatrix} -1 & 2 \\ 0 & -2 \\ 3 & 1 \end{bmatrix}, \quad K^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$L^T K^T = \begin{bmatrix} 3 & 5 & 7 \\ -4 & -8 & -12 \\ 5 & 13 & 21 \end{bmatrix}$$

So, $(KL)^T = L^T K^T$ is verified.

9. Transpose of the product AB :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}, \quad (AB)^T = \begin{bmatrix} 19 & 43 \\ 22 & 50 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 19 & 43 \\ 22 & 50 \end{bmatrix}$$

So, $(AB)^T = B^T A^T$ is verified.

10. Calculate $M \cdot M^T$ and $M^T \cdot M$:

$$M = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$$

$$M \cdot M^T = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 113 & 151 \\ 151 & 181 \end{bmatrix}$$

$$M^T \cdot M = \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} = \begin{bmatrix} 130 & 142 \\ 142 & 164 \end{bmatrix}$$

Both $M \cdot M^T$ and $M^T \cdot M$ are symmetric matrices.

Basic Matrix Multiplication

1. Given matrices

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

find $A \times B$.

2. For matrices

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 4 \\ 3 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 0 & 2 \\ -2 & 3 & 1 \\ 4 & -1 & 3 \end{bmatrix},$$

calculate $A \times B$ and $B \times A$. Are the results equal?

4. Verify that for matrices A and B of order 3×3 , the transpose property $(A \times B)^T = B^T \times A^T$ holds.

5. Calculate the result of $A \times B$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 2 & -3 \\ 3 & -1 & 4 \\ -2 & 5 & 1 \end{bmatrix}.$$

What is special about the result?

6. If A is a diagonal matrix given by $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$,

$$\text{and } B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \text{ find } A \times B \text{ and } B \times A.$$

7. Test if $A \times B = B \times A$ holds for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 4 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}.$$

8. Multiply any 3x3 matrix $A = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 2 & 5 \\ 7 & 0 & 1 \end{bmatrix}$ by the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Confirm that the result is A .

9. Given an invertible 3x3 matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \\ 4 & 1 & 0 \end{bmatrix}$, find A^{-1} and verify that $A \times A^{-1} = I$.

10. If A and B are both invertible matrices, show that $(A \times B)^{-1} = B^{-1} \times A^{-1}$ for matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & -1 \\ 2 & 1 & 3 \\ 1 & 5 & 2 \end{bmatrix}.$$

Inverse up to 3rd order by adjoint method

The **adjoint method** for finding the inverse of a square matrix works by computing the adjoint of the matrix and dividing it by the determinant of the matrix.

Given matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The inverse A^{-1} is given by: $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A),$

where: $\det(A)$ is the determinant of A ,

$\text{adj}(A)$ is the adjoint (transpose of the cofactor matrix).

Find the determinant of A ($\det(A)$):

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Find the cofactor matrix:

The cofactor of an element a_{ij} in A is: $C_{ij} = (-1)^{i+j} \cdot \det(\text{minor}_{ij}),$

where minor_{ij} is the 2×2 submatrix obtained by removing the i -th row and j -th column from A .

For the given matrix A :

$$\text{Cofactor matrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix},$$

where:

$$C_{11} = \det \begin{bmatrix} e & f \\ h & i \end{bmatrix}, \quad C_{12} = -\det \begin{bmatrix} d & f \\ g & i \end{bmatrix}, \quad C_{13} = \det \begin{bmatrix} d & e \\ g & h \end{bmatrix},$$

$$C_{21} = -\det \begin{bmatrix} b & c \\ h & i \end{bmatrix}, \quad C_{22} = \det \begin{bmatrix} a & c \\ g & i \end{bmatrix}, \quad C_{23} = -\det \begin{bmatrix} a & b \\ g & h \end{bmatrix},$$

$$C_{31} = \det \begin{bmatrix} b & c \\ e & f \end{bmatrix}, \quad C_{32} = -\det \begin{bmatrix} a & c \\ d & f \end{bmatrix}, \quad C_{33} = \det \begin{bmatrix} a & b \\ d & e \end{bmatrix}.$$

Find the adjoint of A :

The adjoint of A is the **transpose** of the cofactor matrix: $\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}.$

Compute the inverse: If $\det(A) \neq 0$, the inverse is: $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A).$

If $\det(A) = 0$, the matrix is **singular** and does not have an inverse.

Find the inverse of:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

1. **Compute $\det(A)$:**

$$\begin{aligned}\det(A) &= 2(4 - 1) - 1(2 - 3) + 3(1 - 6) \\ &= 2(3) - 1(-1) + 3(-5) \\ &= 6 + 1 - 15 = -8.\end{aligned}$$

2. **Cofactor matrix:**

$$\text{Cofactor matrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -5 \\ 1 & -5 & 1 \\ -5 & 1 & 3 \end{bmatrix}.$$

3. **Adjoint** (transpose of cofactor matrix): interchange rows and columns of the cofactor matrix.

$$\text{adj}(A) = \begin{bmatrix} 3 & 1 & -5 \\ 1 & -5 & 1 \\ -5 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & -5 \\ 1 & -5 & 1 \\ -5 & 1 & 3 \end{bmatrix}.$$

$$A^{-1} = -\frac{1}{8} \cdot \begin{bmatrix} 3 & 1 & -5 \\ 1 & -5 & 1 \\ -5 & 1 & 3 \end{bmatrix}.$$

Performing scalar multiplication:

$$A^{-1} = \begin{bmatrix} -\frac{3}{8} & -\frac{1}{8} & \frac{5}{8} \\ -\frac{1}{8} & \frac{5}{8} & -\frac{1}{8} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{3}{8} \end{bmatrix}.$$

Find the determinant of the following matrix:

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} .$$

Compute the adjoint of the matrix:

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} .$$

Determine whether the matrix below is invertible:

$$C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} .$$

For the matrix:

$$D = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} ,$$

calculate the cofactor matrix.

Using the adjoint method, find the inverse of:

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} .$$

For what values of x does the following matrix become non-invertible?

$$H = \begin{bmatrix} x & 2 \\ 1 & x+1 \end{bmatrix}.$$

Find the determinant of:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 5 \\ 6 & 3 & 2 \end{bmatrix}.$$

Compute the determinant of:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

For the matrix below, find the cofactor of the element a_{12} :

$$C = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 5 \\ 6 & 3 & 1 \end{bmatrix}.$$

Determine the cofactor matrix of the matrix:

$$E = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & 4 \\ 6 & 8 & 9 \end{bmatrix}.$$

Compute the adjoint of the matrix:

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}.$$

Find the adjoint of:

$$G = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Calculate the inverse of:

$$H = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

If J is a matrix with determinant -10 and adjoint:

$$\text{adj}(J) = \begin{bmatrix} 5 & -2 & 3 \\ 1 & 4 & -1 \\ 2 & 1 & -2 \end{bmatrix},$$

Verify that the inverse of:

$$L = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

is:

$$L^{-1} = \begin{bmatrix} 7 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Determinants

Consider the 3rd order determinant,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This can be expanded along any row or any column. Usually we expand by the 1st row. On expanding along the 1st row

$$|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Minors

Let $A = (a_{ij})$ be a determinant of order n . The minor of the element a_{ij} is the determinant formed by deleting i^{th} row and j^{th} column in which the element belongs and the cofactor of the element is $A_{ij} = (-1)^{i+j} M_{ij}$ where M is the minor of i^{th} row and j^{th} column.

Example 1 Calculate the determinant of the following matrices.

$$(a) \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{pmatrix}$$

Solution

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$
$$= 1(-2 - 12) - 2(-3 - 4) + 3(9 - 2)$$
$$= -14 + 14 + 21 = 21$$

Singular and Non-Singular Matrices:

Definition

A square matrix 'A' is said to be singular if, $|A| = 0$ and it is called non-singular if $|A| \neq 0$.

Note

Only square matrices have determinants.

Example: Find the solution for the matrix $A = \begin{vmatrix} 2 & 4 & 3 \\ 5 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix}$

$$\begin{vmatrix} 2 & 4 & 3 \\ 5 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 \\ 5 & 3 \end{vmatrix} - 4 \begin{vmatrix} 5 & 0 \\ 7 & 3 \end{vmatrix} + 3 \begin{vmatrix} 5 & 1 \\ 7 & 5 \end{vmatrix}$$
$$= 2(3 - 0) - 4(15 - 0) + 3(25 - 7)$$
$$= 6 - 60 + 54 = 0$$

Here $|A| = 0$. So the given matrix is singular

Properties of Determinants (up to 3rd Order)

1. Determinant of a Triangular Matrix:

Statement:

The determinant of a triangular matrix (upper or lower) is the product of its diagonal elements.

Proof:

For an $n \times n$ triangular matrix A :

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

The determinant is computed by cofactor expansion along the first row:

$$\det(A) = a_{11} \cdot \det(A_1),$$

where A_1 is the submatrix obtained by removing the first row and column. Repeating this for each diagonal element, the determinant simplifies to:

$$\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}.$$

Determinant of a Diagonal Matrix

Statement:

The determinant of a diagonal matrix is the product of its diagonal elements.

Definition of a Diagonal Matrix:

A diagonal matrix D is a square matrix where all non-diagonal elements are zero:

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}.$$

Proof:

The determinant of D is calculated using cofactor expansion. For simplicity, we'll demonstrate for a 3×3 diagonal matrix, but the result generalizes for any $n \times n$ diagonal matrix.

For $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$, expand along the first row:

$$\det(D) = d_1 \cdot \det \begin{bmatrix} d_2 & 0 \\ 0 & d_3 \end{bmatrix} - 0 \cdot \det(\dots) + 0 \cdot \det(\dots).$$

The remaining determinant is:

$$\det \begin{bmatrix} d_2 & 0 \\ 0 & d_3 \end{bmatrix} = d_2 \cdot d_3.$$

Substituting back:

$$\det(D) = d_1 \cdot d_2 \cdot d_3.$$

3. Determinant of the Identity Matrix

Statement:

The determinant of the identity matrix is 1.

Proof:

The identity matrix I_n is a diagonal matrix with all diagonal elements equal to 1:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Using property (2):

$$\det(I_n) = 1 \cdot 1 \cdot \dots \cdot 1 = 1.$$

Determinant of a Matrix with Proportional Rows or Columns

Statement:

If two rows (or columns) of a matrix are proportional, then the determinant of the matrix is zero.

Proof:

Definition of Proportional Rows:

Two rows (or columns) of a matrix are proportional if one is a scalar multiple of the other. For a matrix A with rows R_1 and R_2 , this means:

$$R_2 = k \cdot R_1 \quad \text{for some scalar } k.$$

Example (2x2 Matrix):

Let $A = \begin{bmatrix} a & b \\ ka & kb \end{bmatrix}$, where the second row is k times the first row.

The determinant of A is:

$$\det(A) = \begin{vmatrix} a & b \\ ka & kb \end{vmatrix}.$$

Expanding the determinant:

$$\det(A) = (a)(kb) - (b)(ka) = akb - bka = 0.$$

Thus, the determinant is zero.

5. Interchanging Rows or Columns

Statement:

Interchanging two rows (or columns) changes the sign of the determinant.

Proof:

Consider a 2x2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Its determinant is:

$$\det(A) = ad - bc.$$

If we interchange the rows, the matrix becomes:

$$A' = \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

The determinant of A' is:

$$\det(A') = cb - da = -(ad - bc) = -\det(A).$$

For larger matrices, this result generalizes because each interchange reverses the sign of the cofactor expansions.

6. Row or Column Multiplication by a Scalar

Statement:

If a row (or column) is multiplied by a scalar k , the determinant is also multiplied by k .

Proof:

Consider a 2x2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If we multiply the first row by k , the matrix becomes:

$$A' = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}.$$

The determinant of A' is:

$$\det(A') = (ka)(d) - (kb)(c) = k(ad - bc) = k \cdot \det(A).$$

This property extends to higher-order matrices using cofactor expansion.

7. Addition of Rows or Columns

Statement:

Adding a scalar multiple of one row (or column) to another does not change the determinant.

Proof:

Consider a 2x2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Add k times the first row to the second row, giving:

$$A' = \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix}.$$

The determinant of A' is:

$$\det(A') = a(d + kb) - b(c + ka) = ad + akb - bc - bka = ad - bc = \det(A).$$

For higher orders, this follows similarly by expanding along rows or columns.

8. Zero Row or Column

Statement:

If any row (or column) of a matrix is entirely zero, the determinant is zero.

Proof:

For a 2x2 matrix:

$$A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}.$$

The determinant is:

$$\det(A) = 0 \cdot d - 0 \cdot c = 0.$$

This result generalizes to higher orders because every cofactor expansion will involve multiplying by 0.

9. Multiplicative Property

Statement:

For two square matrices A and B of the same order:

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

Properties of determinants

1. The value of a determinant is unaltered by interchanging its rows and columns.

Example

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix} \text{ then}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 1(-2-12) - 2(-3-4) + 3(9-2) \\ &= -14 + 14 + 21 = 21 \end{aligned}$$

Let us interchange the rows and columns of A. Thus we get new matrix A_1 .

Then

$$\begin{aligned}\det(A_1) &= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & -1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} \\ &= 1(-2 - 12) - 3(-2 - 9) + 1(8 - 6) \\ &= -14 + 33 + 2 = 21\end{aligned}$$

Hence $\det(A) = \det(A_1)$.

2. If any two rows (columns) of a determinant are interchanged the determinant changes its sign but its numerical value is unaltered.

Example

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix} \text{ then}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 1(-2 - 12) - 2(-3 - 4) + 3(9 - 2) \\ &= -14 + 14 + 21 = 21 \end{aligned}$$

Let A_1 be the matrix obtained from A by interchanging the first and second row. i.e R_1 and R_2 .

Then

$$\begin{aligned} \det(A_1) &= \begin{vmatrix} 3 & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{vmatrix} = 3 \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 3(-2 - 9) - 2(-1 - 3) + 4(3 - 2) \\ &= -33 + 8 + 4 = -21 \end{aligned}$$

Hence $\det(A) = -\det(A_1)$.

3. If two rows (columns) of a determinant are identical then the value of the determinant is zero.

Example

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & 4 \\ 1 & 1 & -1 \end{bmatrix} \text{ then}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 1 & 3 \\ 3 & 3 & 4 \\ 1 & 1 & -1 \end{vmatrix} = 1 \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} \\ &= 1(-3-4) - 1(-3-4) + 3(3-3) \\ &= -7 + 7 + 0 = 0 \end{aligned}$$

Hence $\det(A) = 0$

4. If every element in a row (or column) of a determinant is multiplied by a constant “k” then the value of the determinant is multiplied by k.

Example

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{bmatrix} \text{ then}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 1(-2 - 12) - 2(-3 - 4) + 3(9 - 2) \\ &= -14 + 14 + 21 = 21 \end{aligned}$$

Let A_1 be the matrix obtained by multiplying the elements of the first row by 2 (ie. here k =2) then

$$\begin{aligned} \det(A_1) &= \begin{vmatrix} 2(1) & 2(2) & 2(3) \\ 3 & 2 & 4 \\ 1 & 3 & -1 \end{vmatrix} = 2 \times 1 \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} - 2 \times 2 \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} + 2 \times 3 \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 2[1(-2 - 12) - 2(-3 - 4) + 3(9 - 2)] \\ &= 2[-14 + 14 + 21] = 2(21) \end{aligned}$$

Hence $\det(A) = 2 \det(A_1)$.

INVERSE OF A MATRIX

Definition

Let A be any square matrix. If there exists another square matrix B Such that $AB = BA = I$ (I is a unit matrix) then B is called the inverse of the matrix A and is denoted by A^{-1} .

The cofactor method is used to find the inverse of a matrix. Using matrices, the solutions of simultaneous equations are found.

Working Rule to find the inverse of the matrix

Step 1: Find the determinant of the matrix.

Step 2: If the value of the determinant is non zero proceed to find the inverse of the matrix.

Step 3: Find the cofactor of each element and form the cofactor matrix.

Step 4: The transpose of the cofactor matrix is the adjoint matrix.

Step 5: The inverse of the matrix $A^{-1} = \frac{adj(A)}{|A|}$

Example

Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

Solution

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

Step 1

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18 - 12) - 1(9 - 3) + (4 - 2) \\ &= 6 - 6 + 2 = 2 \neq 0 \end{aligned}$$

Step 2

The value of the determinant is non zero
 $\therefore A^{-1}$ exists.

Step 3

Let A_{ij} denote the cofactor of a_{ij} in $|A|$

$$A_{11} = \text{Cofactor of } 1 = (-1)^{1+1} \begin{vmatrix} 2 \\ 4 \end{vmatrix}$$

$$\begin{vmatrix} 3 \\ 9 \end{vmatrix} = 18 - 12 = 6$$

$$A_{12} = \text{Cofactor of } 1 = (-1)^{1+3} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 3 \\ 9 \end{vmatrix} = -(9 - 3) = -6$$

$$A_{13} = \text{Cofactor of } 1 = (-1)^{1+3} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 \\ 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_{21} = \text{Cofactor of } 1 = (-1)^{2+1} \begin{vmatrix} 1 \\ 4 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 9 \end{vmatrix} = -(9 - 4) = -5$$

$$A_{22} = \text{Cofactor of } 2 = (-1)^{2+2} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 9 \end{vmatrix} = 9 - 1 = 8$$

$$A_{23} = \text{Cofactor of } 3 = (-1)^{2+3} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 4 \end{vmatrix} = -(4 - 1) = -3$$

$$A_{31} = \text{Cofactor of } 1 = (-1)^{3+1} \begin{vmatrix} 1 \\ 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{32} = \text{Cofactor of } 4 = (-1)^{3+2} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 3 \end{vmatrix} = -(3 - 1) = -2$$

$$A_{33} = \text{Cofactor of } 9 = (-1)^{3+3} \begin{vmatrix} 1 \\ 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 \\ 2 \end{vmatrix} = 2 - 1 = 1$$

Step 4

The matrix formed by cofactors of element of determinant $|A|$ is $\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

$$\therefore \text{adj } A = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Step 5

$$\begin{aligned} A^{-1} &= \frac{\text{adj } A}{|A|} = \frac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & \frac{-5}{2} & \frac{1}{2} \\ -3 & 4 & -1 \\ 1 & \frac{-3}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

SOLUTION OF LINEAR EQUATIONS

Let us consider a system of linear equations with three unknowns

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The matrix form of the equation is $AX=B$ where $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a 3x3 matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Here $AX = B$

Pre multiplying both sides by A^{-1} .

$$(A^{-1}A)X = A^{-1}B$$

We know that $A^{-1}A = A A^{-1} = I$

$$\therefore IX = A^{-1}B$$

since $IX = X$

Hence the solution is $X = A^{-1}B$.

Example

Solve the $x + y + z = 1$, $3x + 5y + 6z = 4$, $9x + 26y + 36z = 16$ by matrix method.

Solution

The given equations are $x + y + z = 1$,

$$3x + 5y + 6z = 4,$$

$$9x + 26y + 36z = 16$$

$$\text{Let } A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 26 & 36 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}$$

The given system of equations can be put in the form of the matrix equation $AX=B$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ 9 & 26 & 36 \end{vmatrix} = 1(180 - 156) - 1(108 - 54) + 1(78 - 45) \\ &= 24 - 54 + 33 = 3 \neq 0 \end{aligned}$$

The value of the determinant is non zero

$\therefore A^{-1}$ exists.

Let A_{ij} ($i, j = 1, 2, 3$) denote the cofactor of a_{ij} in $|A|$

$$A_{11} = \text{Cofactor of } 1 = (-1)^{1+1} \begin{vmatrix} 5 & 6 \\ 26 & 36 \end{vmatrix} = 180 - 156 = 24$$

$$A_{12} = \text{Cofactor of } 1 = (-1)^{1+2} \begin{vmatrix} 3 & 6 \\ 9 & 36 \end{vmatrix} = -(108 - 54) = -54$$

$$A_{13} = \text{Cofactor of } 1 = (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ 9 & 26 \end{vmatrix} = 78 - 45 = 33$$

$$A_{21} = \text{Cofactor of } 3 = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 26 & 36 \end{vmatrix} = -(36 - 26) = -10$$

$$A_{22} = \text{Cofactor of } 5 = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 9 & 36 \end{vmatrix} = 36 - 9 = 27$$

$$A_{23} = \text{Cofactor of } 6 = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 9 & 26 \end{vmatrix} = -(26 - 9) = -17$$

$$A_{31} = \text{Cofactor of } 9 = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 5 & 6 \end{vmatrix} = 6 - 5 = 1$$

$$A_{32} = \text{Cofactor of } 26 = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} = -(6 - 3) = -3$$

$$A_{33} = \text{Cofactor of } 36 = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix} = 5 - 3 = 2$$

The matrix formed by cofactors of element of determinant $|A|$ is $\begin{pmatrix} 24 & -54 & 33 \\ -10 & 27 & -17 \\ 1 & -3 & 2 \end{pmatrix}$

$$\therefore \text{adj } A = \begin{pmatrix} 24 & -10 & 1 \\ -54 & 27 & -3 \\ 33 & -17 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{3} \begin{pmatrix} 24 & -10 & 1 \\ -54 & 27 & -3 \\ 33 & -17 & 2 \end{pmatrix}$$

We Know that $X=A^{-1}B$

$$\begin{aligned}
 \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 24 & -10 & 1 \\ -54 & 27 & -3 \\ 33 & -17 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix} \\
 &= \frac{1}{3} \begin{pmatrix} 24 \cdot 1 + (-10) \cdot 4 + 1 \cdot 16 \\ (-54) \cdot 1 + 27 \cdot 4 + (-3) \cdot 16 \\ 33 \cdot 1 + (-17) \cdot 4 + 2 \cdot 16 \end{pmatrix} \\
 &= \frac{1}{3} \begin{pmatrix} 0 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}
 \end{aligned}$$

$$\mathbf{x = 0, y = 2, z = -1.}$$

What is a Function?

A function relates an input to an output.



It is like a machine that has an input and an output.

And the output is related somehow to the input.

$f(x)$

" $f(x) = \dots$ " is the classic way of writing a function.
And there are other ways, as you will see!

Input, Relationship, Output

We will see many ways to think about functions, but there are always three main parts:

- The input
- The relationship
- The output

A function is a rule or a relationship that assigns to each input exactly one output.

Formal Definition

A function f from a set A (called the domain) to a set B (called the codomain) is written as:

$$f : A \rightarrow B$$

For each element $x \in A$, the function assigns a unique element $f(x) \in B$.

1. **Domain:** The set of all possible inputs for the function.
2. **Codomain:** The set in which the outputs of the function lie.
3. **Range:** The actual set of outputs the function produces.

Some Examples of Functions

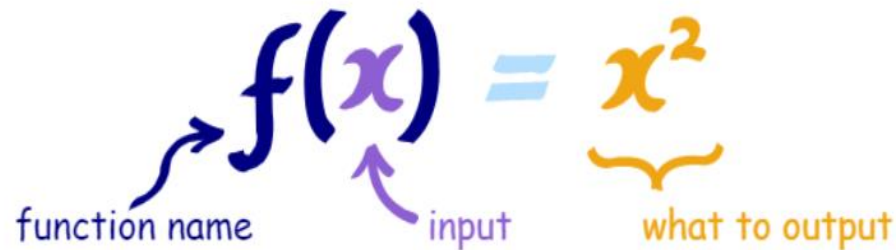
- x^2 (squaring) is a function
- x^3+1 is also a function
- Sine, Cosine and Tangent are functions used in trigonometry
- and there are lots more!

Names

First, it is useful to give a function a **name**.

The most common name is "**f**", but we can have other names like "**g**" ... or even "**marmalade**" if we want.

But let's use "f":



The diagram shows the equation $f(x) = x^2$. The 'f' is in dark blue, the 'x' is in purple, and the 'x^2' is in orange. A blue arrow points from the text 'function name' to the 'f'. A purple arrow points from the text 'input' to the 'x'. A blue bracket is under the 'x^2', with the text 'what to output' below it.

We say "f of x equals x squared"

what goes **into** the function is put inside parentheses () after the name of the function:

So **f(x)** shows us the function is called "**f**", and "**x**" goes **in**

And we usually see what a function does with the input:

f(x) = x² shows us that function "**f**" takes "**x**" and squares it.

Example: with $f(x) = x^2$:

- an input of 4
- becomes an output of 16.

In fact we can write $f(4) = 16$.

$$f(x) = 1 - x + x^2$$

$$f(2) = 1 - 2 + 2^2 = 3$$

Let $f(x) = x^2$, where the domain is all real numbers ($\mathbb{R}$).

$$\text{If } x = 2, \text{ then } f(2) = 2^2 = 4.$$

$$\text{If } x = -3, \text{ then } f(-3) = (-3)^2 = 9.$$

Here:

Domain: All real numbers ($\mathbb{R}$).

Codomain: All real numbers ($\mathbb{R}$).

Range: Non-negative real numbers ($[0, \infty)$).

Example: "Multiply by 2" is a very simple function.

Here are the three parts:

Input	<i>Relationship</i>	Output
0	$\times 2$	0
1	$\times 2$	2
7	$\times 2$	14
10	$\times 2$	20
...	...	...

For an input of 50, what is the output?



Example: this tree grows 20 cm every year, so the height of the tree is **related** to its age using the function ***h***:

$$h(\text{age}) = \text{age} \times 20$$

So, if the age is 10 years, the height is:

$$h(10) = 10 \times 20 = 200 \text{ cm}$$

Here are some example values:

age	$h(\text{age}) = \text{age} \times 20$
0	0
1	20
3.2	64
15	300
...	...

Definition

A relation R from a set A to a set B is a subset of the Cartesian product $A \times B$. This means R is a set of ordered pairs (a, b) , where $a \in A$ and $b \in B$.

Notation:

If $(a, b) \in R$, we say a is related to b , and write $a R b$.

Example: "is greater than" can be a relation between two numbers a and b .

Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$.

The Cartesian product $A \times B$ is the set of all possible ordered pairs where the first element is from A , and the second is from B :

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}.$$

The relation R is a subset of $A \times B$, meaning it includes only some of these pairs.

The given relation R is: $R = \{(1, 4), (2, 5), (3, 4)\}.$

This means: 1 is related to 4, 2 is related to 5, 3 is related to 4.

A relation is a set of ordered pairs.

The domain is the set of all x values in the relation

$$\text{domain} = \{-1, 0, 2, 4, 9\}$$

These are the x values written in a set from smallest to largest

$$\{(2, 3), (-1, 5), (4, -2), (9, 9), (0, -6)\}$$

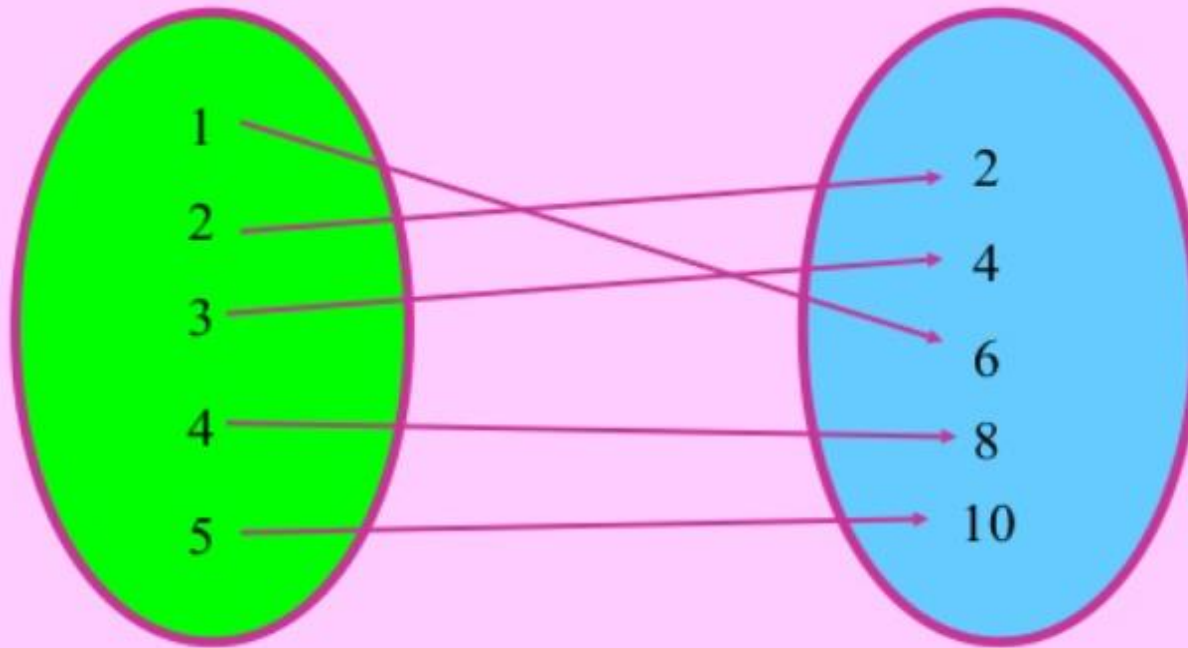
← This is a relation

These are the y values written in a set from smallest to largest

$$\text{range} = \{-6, -2, 3, 5, 9\}$$

The range is the set of all y values in the relation

A relation assigns the x's with y's

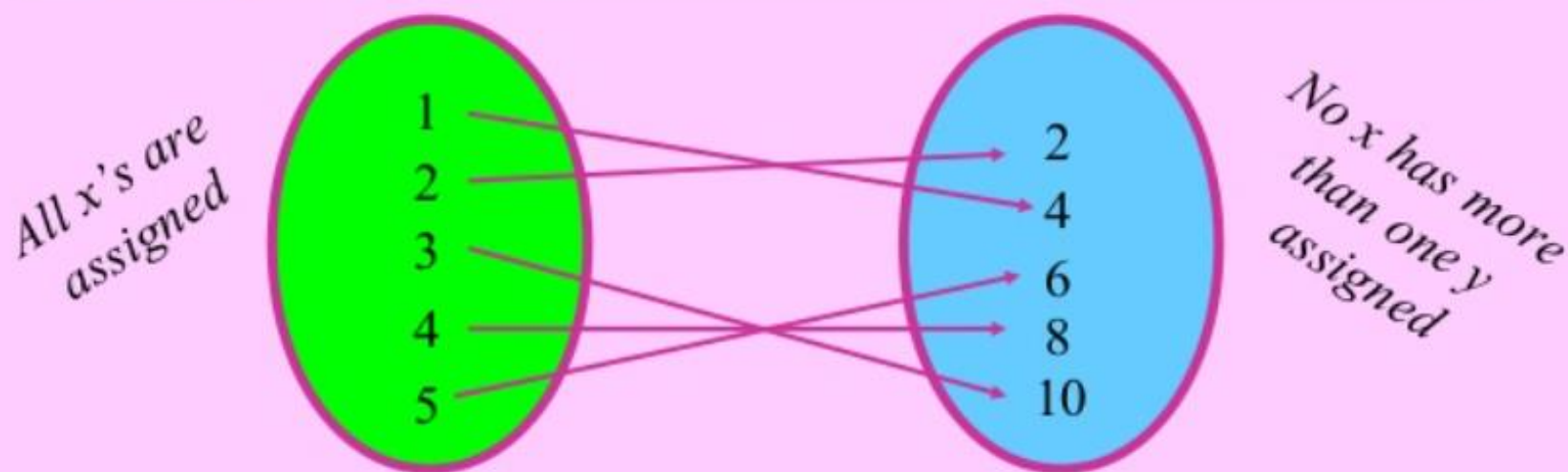


Domain (set of all x's)

Range (set of all y's)

This relation can be written $\{(1,6), (2,2), (3,4), (4,8), (5,10)\}$

A function f from set A to set B is a rule of correspondence that assigns to each element x in the set A **exactly one element y** in the set B.



***This is a function
---it meets our
conditions***

Must use all the x 's

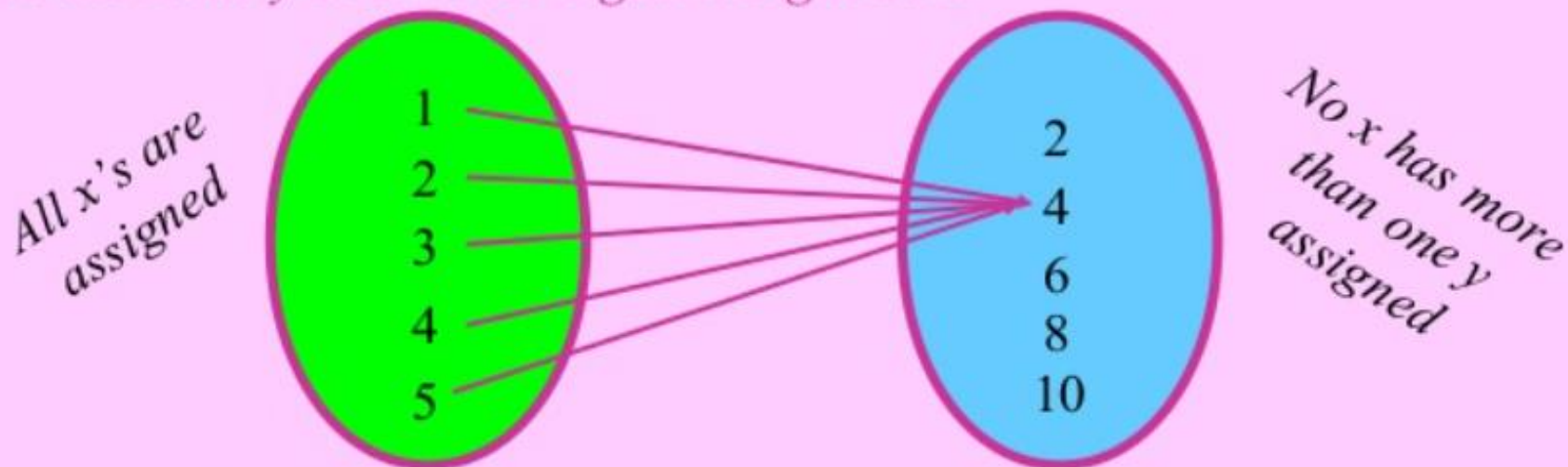


The x value can only be assigned to one y



Let's look at another relation and decide if it is a function.

The second condition says each x can have only one y , but it CAN be the same y as another x gets assigned to.



Set A is the domain

Set B is the range

This is a function
---it meets our
conditions

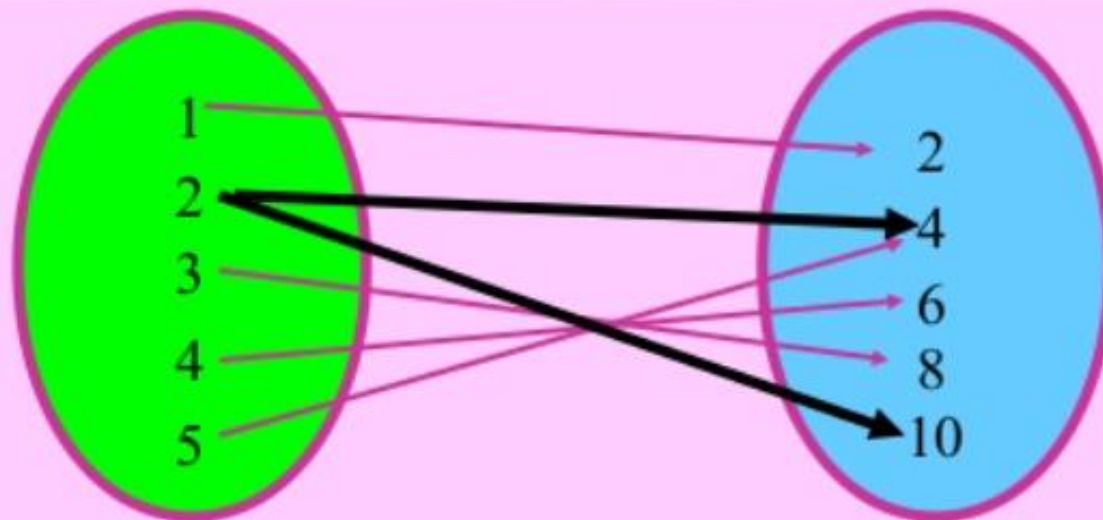
Must use all the x 's



The x value can only be assigned to one y



A good example that you can “relate” to is students in our maths class this semester are set A. The grade they earn out of the class is set B. Each student must be assigned a grade and **can only be assigned ONE grade**, but more than one student can get the same grade (we hope so---we want lots of A’s). The example shown on the previous screen had each student getting the same grade. That’s okay.



2 was assigned both 4 and 10

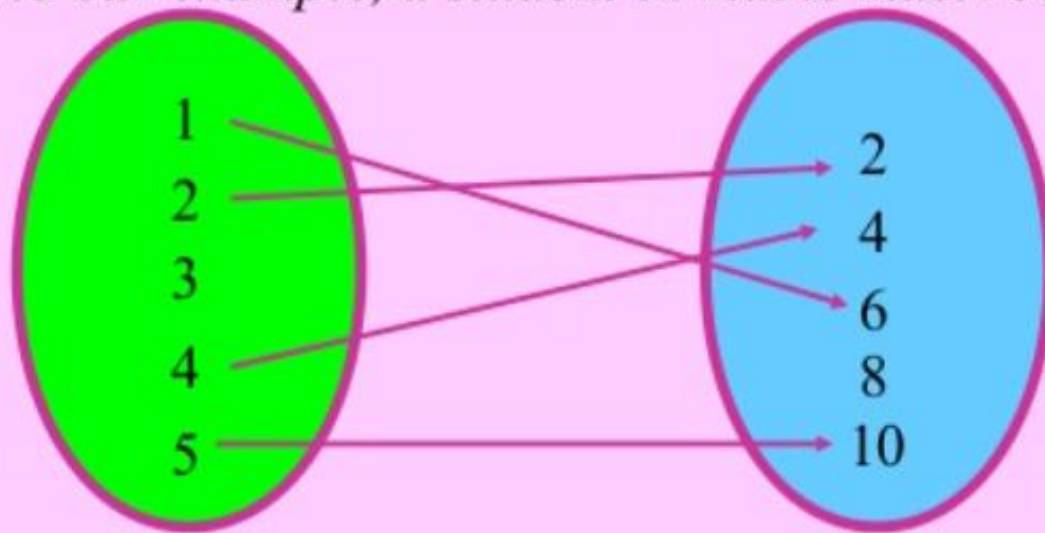
Is the relation shown above a function? **NO**

Why not???

Check this relation out to determine if it is a function.

It is not---3 didn't get assigned to anything

Comparing to our example, a student in maths must receive a grade



Set A is the domain

Set B is the range

***This is not a
function---it
doesn't assign
each x with a y***

Must use all the x's

The x value can only be assigned to one y

Function Notation

We commonly call functions by letters. Because function starts with f , it is a commonly used letter to refer to functions.

$$f(x) = 2x^2 - 3x + 6$$

This means
the right
hand side is
a function
called f

This means
the right hand
side has the
variable x in it

*The left side DOES NOT
MEAN f times x like
brackets usually do, it
simply tells us what is on
the right hand side.*

The left hand side of this equation is the function notation. It tells us two things. We called the function f and the variable in the function is x .

Remember---this tells you what is on the right hand side---it is not something you work. It says that the right hand side is the function f and it has x in it.

$$f(x) = 2x^2 - 3x + 6$$
$$f(2) = 2(2)^2 - 3(2) + 6$$

$$f(2) = 2(4) - 3(2) + 6 = 8 - 6 + 6 = 8$$

So we have a function called f that has the variable x in it.

Using function notation we could then ask the following:

Find $f(2)$.

This means to find the function f and instead of having an x in it, put a 2 in it. So let's take the function above and make brackets everywhere the x was and in its place, put in a 2.

Don't forget order of operations---powers, then multiplication, finally addition & subtraction

Find $f(-2)$.

$$f(x) = 2x^2 - 3x + 6$$


$$f(-2) = 2(-2)^2 - 3(-2) + 6$$

$$f(-2) = 2(4) - 3(-2) + 6 = 8 + 6 + 6 = 20$$

This means to find the function f and instead of having an x in it, put a -2 in it. So let's take the function above and make brackets everywhere the x was and in its place, put in a -2 .

Example

- Given the function $f(x) = 3x + 1$ find
- a) $f(2)$
- b) $f(-1)$
- c) $f(6)$

Solution:

- a) The output from the function needs to be found when the argument or input is 2. We need to replace x by 2 in the expression for the function. We find, $f(2) = 3(2) + 1 = 6 + 1 = 7$.
- b) Here the argument is -1 . We find, $f(-1) = 3(-1) + 1 = -3 + 1 = -2$.
- c) Here, $f(6) = 3(6) + 1 = 18 + 1 = 19$.

Exercise

- Given the function defined by $f(x) = \frac{1}{2}x - 1$ find the function values.

a) $f(0)$

b) $f(2)$

c) $f(-2)$

d) $f(4)$

Answer:

a) -1

b) 0

c) -2

d) 1

- Given the function defined by , $g(x) = -2x - 3$ find the function values.

a) $g(1)$

b) $g(0)$

c) $g(-3)$

d) $g\left(\frac{1}{2}\right)$

Answer:

a) -5

b) -3

c) 3

d) -4

- Given the function $f(x) = 3x + 2$ find
- a) $f(t)$
- b) $f(2t)$
- c) $f(z + 2)$
- d) $f(-5x)$

Solution:

$$a) f(t) = 3t + 2$$

$$b) f(2x) = 3(2t) + 2 = 6t + 2$$

$$c) f(z + 2) = 3(z + 2) + 2 = 3z + 6 + 2 = 3z + 8$$

$$d) f(-5x) = 3(-5x) + 2 = -15x + 2$$

Exercise

- Given the functions defined by $f(x) = x^2 - 2x$ and $g(x) = 3x + 5$ find the function values.

- a) $f(t)$
- b) $g(w + 4)$
- c) $f(-t)$
- d) $g(-5w)$

Answer:

- a) $t^2 - 2t$
- b) $3w + 17$
- c) $t^2 + 2t$
- d) $-15w + 5$

Arithmetic Operations on Functions

- **Sum:** $(f + g)(x) = f(x) + g(x)$, domain: the intersection of the domains of f and g .
- **Difference:** $(f - g)(x) = f(x) - g(x)$, domain: the intersection of the domains of f and g .
- **Product:** $(f \times g)(x) = f(x) \times g(x)$, domain: the intersection of the domains of f and g .
- **Quotient:** $(f \div g)(x) = f(x) \div g(x)$, domain: the intersection of the domains of f and g with the points where $g(x) = 0$ excluded.

Example

Given $f(x) = 3x - 1$ and $g(x) = x^2 - 2x - 3$

1. $(f + g)x = f(x) + g(x) = (3x - 1) + (x^2 - 2x - 3) = 3x - 1 + x^2 - 2x - 3 = x^2 + x - 4$
2. $(f - g)x = f(x) - g(x) = (3x - 1) - (x^2 - 2x - 3) = 3x - 1 - x^2 + 2x + 3 = -x^2 + 5x + 2$
3. $(f \times g)x = f(x) \times g(x) = (3x - 1) \times (x^2 - 2x - 3) = 3x(x^2 -$

- Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 1$, Find $(f \circ g)(2)$.

Solution:

- $(f \circ g)(x) = f(g(x)) = f(3x - 1) = (3x - 1)^2 + 2(3x - 1)$
- $(f \circ g)(x) = (3x - 1)(3x - 1) + 2(3x - 1)$
- $(f \circ g)(x) = 9x^2 - 3x - 3x + 1 + 6x - 2$
- $(f \circ g)(x) = 9x^2 - 1$

So,

- $(f \circ g)(2) = 9(2)^2 - 1 = 9 \times 4 - 1 = 36 - 1 = 35$

- Let $f(x) = x^2 + 1$ and $g(x) = x - 1$. Then show that $f \circ g \neq g \circ f$.

Solution:

- $f \circ g(x) = f(g(x)) = f(x - 1) = (x - 1)^2 + 1$
- $f \circ g(x) = x^2 - 2x + 1 + 1 = x^2 - 2x + 2$
- $f \circ g(x) = x^2 - 2x + 2$

And

- $g \circ f(x) = g(f(x)) = g(x^2 + 1) = (x^2 + 1) - 1$
- $g \circ f(x) = x^2 + 1 - 1 = x^2$
- $g \circ f(x) = x^2$

Hence $f \circ g \neq g \circ f$

A function, $f(x)$, is an *even function* if $f(x) = f(-x)$, for all x in its domain.

- **Show that $f(x) = x^2 - 3$ is an even function.**

Solution:

- Lets use numerical values $x = 3$ and $x = -3$.

Evaluate $f(3)$ in $f(x)$;

$$f(x) = x^2 - 3 \text{ and}$$

$$f(3) = (3)^2 - 3$$

$$f(3) = 6$$

Evaluate $f(-3)$ in $f(x)$;

$$f(x) = x^2 - 3$$

$$f(-3) = (-3)^2 - 3$$

$$f(-3) = 6$$

$$\text{Now, } f(-3) = f(3)$$

Hence, $f(x) = x^2 - 3$ is an even function.

Show that the given Functions are Even:

1. $f(x) = -3x^2 + 4$

2. $g(x) = 3/x^2 + 2$

3. $p(x) = x^6 + 4x^2 - 1$

4. $h(x) = x^2 + 5$

5. $k(x) = |3x|$

6. $z(x) = x^3 \sin x$

Odd Function

A function, $f(x)$, is an *odd function* if $f(-x) = -f(x)$, for all x in its domain.

- **Example: Show that $f(x) = x^3 - 4x$ is an odd function.**
- Let's use numerical values $x = 4$ and $x = -4$.

Evaluate $f(4)$ in $f(x)$;

$$f(x) = x^3 - 4x \text{ and}$$

$$f(4) = (4)^3 - 4(4)$$

$$f(4) = 48$$

$$\text{Now, } f(-4) = -f(4)$$

Hence, $f(x) = x^3 - 4x$ is an odd function.

Evaluate $f(-4)$ in $f(x)$;

$$f(x) = x^3 - 4x$$

$$f(-4) = (-4)^3 - 4(-4)$$

$$f(-4) = -48$$

Limit of a function

Basic Notation

The limit of a function $f(x)$ as x approaches a is written as:

$$\lim_{x \rightarrow a} f(x)$$

It means we are analyzing how $f(x)$ behaves as x gets arbitrarily close to a .

Example

Evaluate:

$$\lim_{x \rightarrow 3} (2x + 5)$$

Substitute $x = 3$ into the function:

$$2x + 5 = 2(3) + 5 = 6 + 5 = 11$$

Final Answer:

$$\lim_{x \rightarrow 3} (2x + 5) = 11$$

Limit of a Function

Suppose that the function $f(x)$ is defined for all values of x near a , but not necessarily at a . If as x approaches a (without actually attaining the value a), $f(x)$ approaches the number L , then we say that L is the limit of $f(x)$ as x approaches a , and write

$$\lim_{x \rightarrow a} f(x) = L$$

Types of Limits

Finite Limit: If $\lim_{x \rightarrow a} f(x) = L$, the value L is the finite limit of $f(x)$ as $x \rightarrow a$.

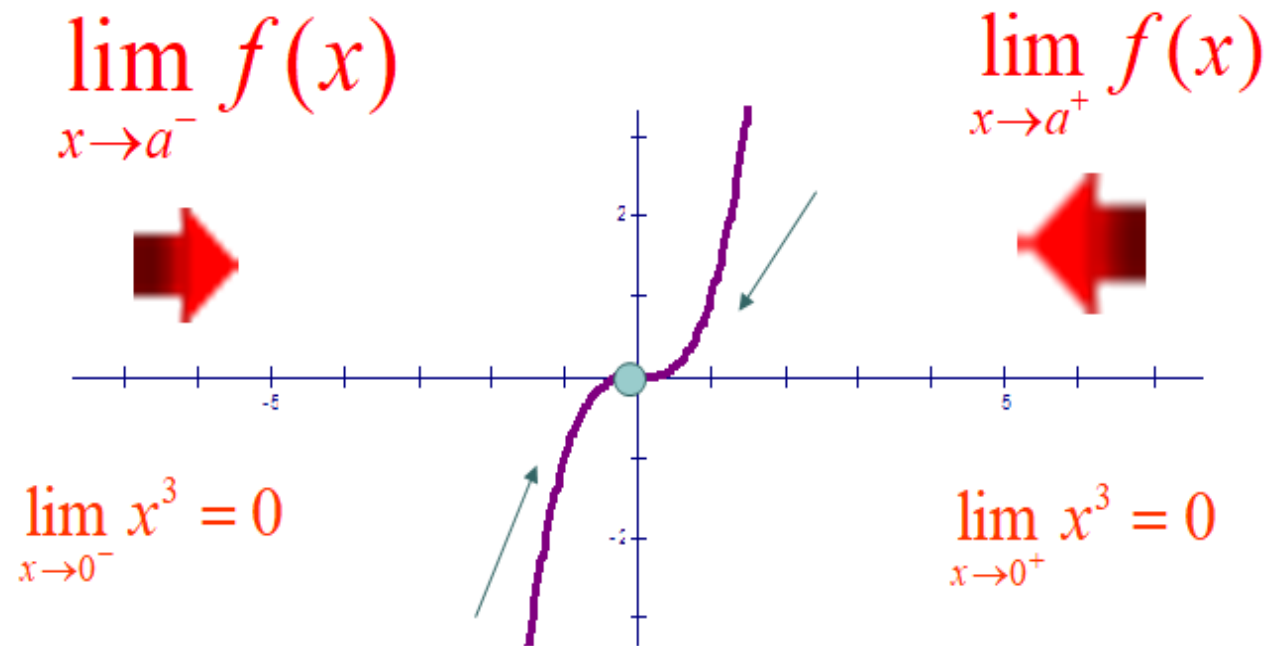
Infinite Limit: If $f(x)$ increases or decreases without bound as $x \rightarrow a$, we write $\lim_{x \rightarrow a} f(x) = \infty$ or $\lim_{x \rightarrow a} f(x) = -\infty$.

One-sided Limits:

Left-hand limit: $\lim_{x \rightarrow a^-} f(x)$ (as x approaches a from the left).

Right-hand limit: $\lim_{x \rightarrow a^+} f(x)$ (as x approaches a from the right).

Left & Right Hand Limits



Limits (*An Introduction*)

Approaching ...

Sometimes we can't work something out directly ... but we **can** see what it should be as we get closer and closer!

Example:

$$\frac{(x^2 - 1)}{(x - 1)}$$

Let's work it out for $x=1$:

$$\frac{(1^2 - 1)}{(1 - 1)} = \frac{(1 - 1)}{(1 - 1)} = \frac{0}{0}$$

Now $0/0$ is a difficulty! We don't really know the value of $0/0$ (it is "indeterminate"), so we need another way of answering this.

So instead of trying to work it out for $x=1$ let's try **approaching** it closer and closer:

Example Continued:

x	$\frac{(x^2 - 1)}{(x - 1)}$
0.5	1.50000
0.9	1.90000
0.99	1.99000
0.999	1.99900
0.9999	1.99990
0.99999	1.99999
...	...

Now we see that as x gets close to 1, then $\frac{(x^2-1)}{(x-1)}$ gets **close to 2**

Example Continued

So, let's try from the other side:

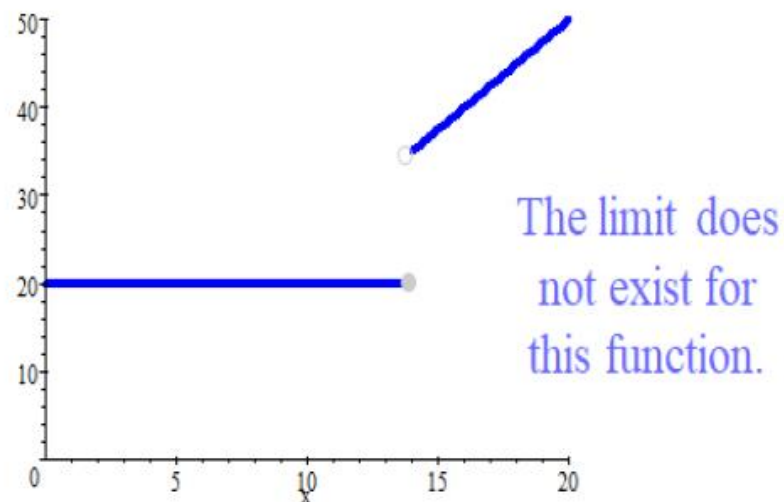
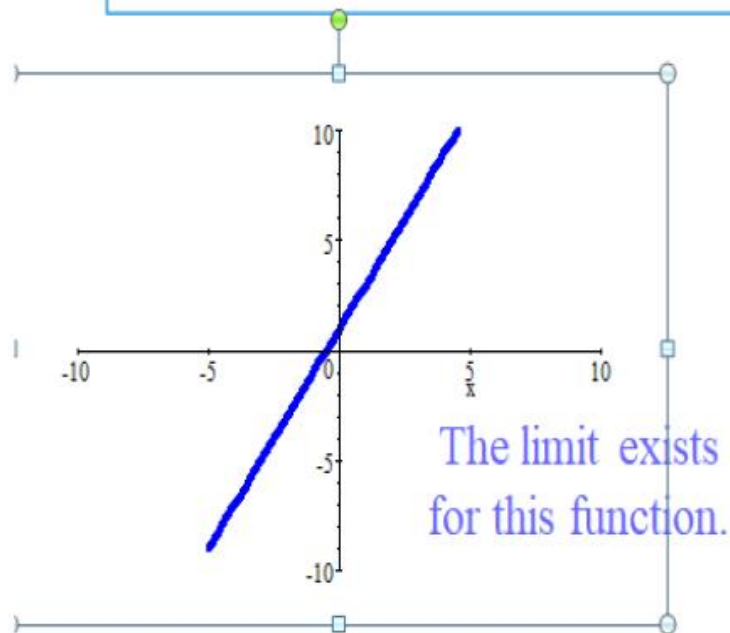
x	$\frac{(x^2 - 1)}{(x - 1)}$
1.5	2.50000
1.1	2.10000
1.01	2.01000
1.001	2.00100
1.0001	2.00010
1.00001	2.00001
...	...

Also heading for 2, so that's OK

THEOREM

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$



When it is different from different sides

How about a function $f(x)$ with a "break" in it like this:

The limit does not exist at "a"

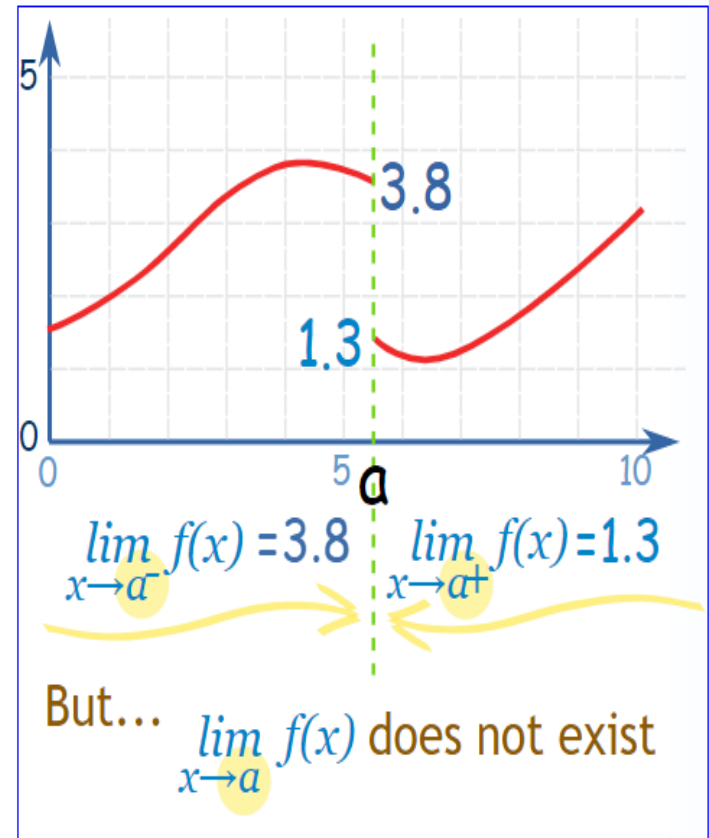
We can't say what the value at "a" is, because there are two competing answers:

- 3.8 from the left, and
- 1.3 from the right

But we **can** use the special "-" or "+" signs (as shown) to define one sided limits:

- the **left-hand** limit (-) is 3.8
- the **right-hand** limit (+) is 1.3

And the ordinary limit **"does not exist"**



Rules of Limits

Sum Rule: $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$

Product Rule: $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$

Quotient Rule: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)},$ provided $g(x) \neq 0.$

Constant Rule: $\lim_{x \rightarrow a} c = c,$ where c is a constant.

Power Rule: $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n,$ where n is a positive integer.

Sum Rule

If you have two functions $f(x)$ and $g(x)$:

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

Example:

$$\begin{aligned}\lim_{x \rightarrow 2} (x^2 + 3x) &= \lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 3x \\ &= 2^2 + 3(2) = 4 + 6 = 10\end{aligned}$$

Product Rule

For the product of two functions:

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

Example:

$$\begin{aligned}\lim_{x \rightarrow 2} (x \cdot (x + 1)) &= \lim_{x \rightarrow 2} x \cdot \lim_{x \rightarrow 2} (x + 1) \\ &= 2 \cdot (2 + 1) = 2 \cdot 3 = 6\end{aligned}$$

Quotient Rule

For the quotient of two functions:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad (\text{if } \lim_{x \rightarrow a} g(x) \neq 0)$$

Example:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2}{x - 1} &= \frac{\lim_{x \rightarrow 3} x^2}{\lim_{x \rightarrow 3} (x - 1)} \\ &= \frac{3^2}{3 - 1} = \frac{9}{2} \end{aligned}$$

Constant Rule

The limit of a constant c is simply c :

$$\lim_{x \rightarrow a} c = c$$

Example:

$$\lim_{x \rightarrow 5} 7 = 7$$

Power Rule

If $f(x)$ has a limit, then:

$$\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$$

Example:

$$\lim_{x \rightarrow 2} (x^3) = (\lim_{x \rightarrow 2} x)^3 = 2^3 = 8$$

Root Rule

If n is a positive integer, and $\lim_{x \rightarrow a} f(x) \geq 0$:

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

Example:

$$\lim_{x \rightarrow 4} \sqrt{x} = \sqrt{\lim_{x \rightarrow 4} x} = \sqrt{4} = 2$$

Scalar Multiplication Rule

If c is a constant:

$$\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$$

Example:

$$\lim_{x \rightarrow 2} 3x = 3 \cdot \lim_{x \rightarrow 2} x = 3 \cdot 2 = 6$$

Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ near a , and:

$$\lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} g(x) = L,$$

then:

$$\lim_{x \rightarrow a} f(x) = L$$

Infinity Rule

As $x \rightarrow \infty$, the reciprocal of a growing term approaches 0:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Finding a Limit of a Linear Function

Example Find $\lim_{x \rightarrow 4} (3 + 2x)$.

Solution $\lim_{x \rightarrow 4} (3 + 2x) = \lim_{x \rightarrow 4} 3 + \lim_{x \rightarrow 4} 2x$

$$= 3 + \lim_{x \rightarrow 4} 2 \cdot \lim_{x \rightarrow 4} x \quad \text{Rules 1 and 4}$$

$$= 3 + 2 \cdot 4 \quad \text{Rules 1 and 2}$$

$$= 11$$

Example Find $\lim_{x \rightarrow 5} 3x^2$.

Solution

$$\begin{aligned}\lim_{x \rightarrow 5} 3x^2 &= \lim_{x \rightarrow 5} 3 \cdot \lim_{x \rightarrow 5} x^2 \\&= 3 \cdot \lim_{x \rightarrow 5} x^2 \\&= 3 \cdot \lim_{x \rightarrow 5} x \cdot \lim_{x \rightarrow 5} x \\&= 3 \cdot 5 \cdot 5 \\&= 75\end{aligned}$$

Rational fraction rule

The **rational function rule** applies when you evaluate the limit of a rational function.

A rational function is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

Case 1: Direct Substitution Works

If substituting $x = a$ into $P(x)$ and $Q(x)$ does not produce an indeterminate form (like $\frac{0}{0}$), you can directly substitute:

$$\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

Example:

$$\lim_{x \rightarrow 1} \frac{x^2 + 3}{x + 2}$$

Substitute $x = 1$:

$$\frac{1^2 + 3}{1 + 2} = \frac{4}{3}.$$

So:

$$\lim_{x \rightarrow 1} \frac{x^2 + 3}{x + 2} = \frac{4}{3}.$$

Case 2: Indeterminate Form ($\frac{0}{0}$)

If direct substitution produces $\frac{0}{0}$, factorize or simplify the function to remove the indeterminate form, then reapply the limit.

Example: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Substitute $x = 2$, and you get $\frac{0}{0}$. Factorize:

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}.$$

$$\frac{\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} = x + 2.$$

Now substitute $x = 2$: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 2 + 2 = 4.$

Case 3: Limits at Infinity

For a rational function as $x \rightarrow \infty$ or $x \rightarrow -\infty$, compare the degrees of $P(x)$ and $Q(x)$:

1. Degree of $P(x) < Q(x)$:

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = 0.$$

Example: $\lim_{x \rightarrow \infty} \frac{x}{x^2 + 1} = 0$ (since the denominator grows faster).

2. Degree of $P(x) = Q(x)$:

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \frac{\text{leading coefficient of } P(x)}{\text{leading coefficient of } Q(x)}.$$

Example: $\lim_{x \rightarrow \infty} \frac{3x^2 + 2}{5x^2 + 7} = \frac{3}{5}.$

3. Degree of $P(x) > Q(x)$:

$$\lim_{x \rightarrow \infty} \frac{P(x)}{Q(x)} = \infty \quad (\text{or } -\infty \text{ depending on the signs}).$$

Example: $\lim_{x \rightarrow \infty} \frac{x^3 + 1}{x^2 + 7} = \infty.$

Rules for Limits (Continued)

Exponent rule For any real number $b > 0$,

$$\lim_{x \rightarrow a} b^{f(x)} = b^{\lim_{x \rightarrow a} f(x)}.$$

Logarithm rule For any real number $b > 0$ with $b \neq 1$,

$$\lim_{x \rightarrow a} [\log_b f(x)] = \log_b \left[\lim_{x \rightarrow a} f(x) \right]$$

provided that $\lim_{x \rightarrow a} f(x) > 0$.

Example Find $\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - 3x + 2}$.

Solution Rule 7 cannot be applied directly since the denominator is 0. First factor the numerator and denominator

$$\frac{x^2 + 2x - 3}{x^2 - 3x + 2} = \frac{(x+3)(x-1)}{(x-2)(x-1)} = \frac{x+3}{x-2}$$

$$= \lim_{x \rightarrow 1} \frac{x+3}{x-2} = \frac{1+3}{1-2} = -4$$

Example Find $\lim_{x \rightarrow \infty} \frac{5x^2 - 7x + 1}{2x^2 + x + 5}$.

Solution Divide numerator and denominator by the highest power of x involved, x^2 .

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{5x^2 - 7x + 1}{2x^2 + x + 5} &= \lim_{x \rightarrow \infty} \frac{5 - \frac{7}{x} + \frac{1}{x^2}}{2 + \frac{1}{x} + \frac{5}{x^2}} = \lim_{x \rightarrow \infty} \frac{5 - \frac{7}{x} + \frac{1}{x^2}}{2 + \frac{1}{x} + \frac{5}{x^2}} \\&= \frac{\lim_{x \rightarrow \infty} 5 - \frac{7}{x} + \frac{1}{x^2}}{\lim_{x \rightarrow \infty} 2 + \frac{1}{x} + \frac{5}{x^2}} = \frac{\lim_{x \rightarrow \infty} 5 - 7 \cdot \lim_{x \rightarrow \infty} \frac{1}{x} + \lim_{x \rightarrow \infty} \frac{1}{x^2}}{\lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} \frac{1}{x} + 5 \cdot \lim_{x \rightarrow \infty} \frac{1}{x^2}} \\&= \frac{5 - 0 + 0}{2 + 0 + 0} = \frac{5}{2}\end{aligned}$$

$\therefore$

limit rules for trigonometric functions (sin, cos, tan)

1. Fundamental Limits

$$(a) \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \quad (b) \lim_{x \rightarrow 0} \frac{\tan(x)}{x} = 1 \quad (c) \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x} = 0$$

2. Derived Limits

$$(a) \lim_{x \rightarrow 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}, \text{ where } a, b > 0. \quad (b) \lim_{x \rightarrow 0} \frac{\tan(ax)}{\sin(bx)} = \frac{a}{b},$$

where $a, b > 0$.

Fundamental Limits

$$(a) \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$(b) \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln(a), \text{ where } a > 0$$

$$(c) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$(d) \lim_{x \rightarrow 0} (1 + x)^{1/x} = e$$

Limits Involving Exponential Growth

$$(a) \lim_{x \rightarrow \infty} e^x = \infty$$

$$(b) \lim_{x \rightarrow -\infty} e^x = 0$$

$$(c) \lim_{x \rightarrow \infty} a^x = \infty, \text{ where } a > 1$$

$$(d) \lim_{x \rightarrow -\infty} a^x = 0, \text{ where } a > 1$$

Special Cases

$$(a) \lim_{x \rightarrow 0} e^{kx} = 1, \text{ for any constant } k$$

$$(b) \lim_{x \rightarrow 0} \frac{e^{kx} - 1}{x} = k$$

8. कुछ विशिष्ट व्यंजकों की सीमाएँ

$$1. \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = \lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \rightarrow 0} \left(\frac{\tan}{x} \right) = 1,$$

$$2. \lim_{x \rightarrow \infty} \left(x + \frac{1}{x} \right)^x = \lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$3. \lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^x = e^a$$

$$4. \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \quad (n \neq 1)$$

$$5. \lim_{x \rightarrow \infty} a^x = \begin{cases} \infty & \text{यदि } a > 1 \\ 0 & \text{यदि } a < 1 \end{cases}$$

$$6. \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = 1$$

$$7. \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x} \right) = \log_e a, \quad a > 0$$

$$8. \lim_{x \rightarrow 0} (1+ax)^{1/a} = e^a$$

$$9. \lim_{x \rightarrow a} \cos x = 1$$

9. कुछ उपयोगी विस्तार

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \infty$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad |x| < 1$$

$$a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \dots$$

$$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_n x^0 a^n$$

$$(1+x)^n = 1 + n \cdot x + \frac{n(n-1)}{2} x^2 + \dots \quad (x < 1)$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \dots$$

$$\cos^{-1} x = x - \frac{x^3}{6} + \dots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

उदाहरण $\frac{3x^2 - 4x + 1}{x^2 - 4x + 3}$ की सीमा $x \rightarrow 1$ के लिए ज्ञात करो।

हल—हम देखते हैं कि फलन $\frac{3x^2 - 4x + 1}{x^2 - 4x + 3}$ में $x = 1$ लिखने पर अंश और हर दोनों ही शून्य हो जाते हैं

अतः x की जगह $1 + h$ लिखने पर,

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{3x^2 - 4x + 1}{x^2 - 4x + 3} &= \lim_{h \rightarrow 0} \frac{3(1+h)^2 - 4(1+h) + 1}{(1+h)^2 - 4(1+h) + 3} \\&= \lim_{h \rightarrow 0} \frac{3(1+2h+h^2) - 4 - 4h + 1}{1+2h+h^2 - 4 - 4h + 3} \\&= \lim_{h \rightarrow 0} \frac{3+6h+3h^2 - 4 - 4h + 1}{h^2 - 2h} \\&= \lim_{h \rightarrow 0} \frac{3h^2 + 2h}{h^2 - 2h} \\&= \lim_{h \rightarrow 0} \frac{h(3h+2)}{h(h-2)} \\&= \lim_{h \rightarrow 0} \frac{3h+2}{h-2} \\&= \frac{0+2}{0-2} = -1\end{aligned}$$

नोट— $x = 1 - h$ रखने पर भी वामावर्ती सीमा का मान यही आता है।

दूसरी विधि

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{3x^2 - 4x + 1}{x^2 - 4x + 3} &= \lim_{x \rightarrow 1} \frac{(3x-1)(x-1)}{(x-3)(x-1)} \\&= \lim_{x \rightarrow 1} \frac{3x-1}{x-3} = \frac{3(1)-1}{1-3} = -1\end{aligned}$$

Evaluate

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$$

Solution:

Using the standard limit property $\lim_{x \rightarrow 0} \frac{\sin(kx)}{x} = k$:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} = 3$$

Evaluate

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 7x + 3}{2x^2 + 4x + 1}$$

Solution:

Divide numerator and denominator by x^2 (highest power of x in the denominator):

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 7x + 3}{2x^2 + 4x + 1} = \lim_{x \rightarrow \infty} \frac{5 + \frac{7}{x} + \frac{3}{x^2}}{2 + \frac{4}{x} + \frac{1}{x^2}}$$

As $x \rightarrow \infty$, $\frac{7}{x} \rightarrow 0$ and $\frac{3}{x^2} \rightarrow 0$, so:

$$\frac{5 + 0 + 0}{2 + 0 + 0} = \frac{5}{2}$$

L hospital rule

L'Hôpital's Rule is a mathematical tool used to evaluate the limits of indeterminate forms such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The rule states:

The Rule:

lf: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ results in an indeterminate form, e.g., $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

then:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

provided that:

1. $f'(x)$ and $g'(x)$ exist, and
2. $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ exists or approaches infinity.

Prove that

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

Solution:

Using L'Hôpital's Rule (indeterminate form $\frac{0}{0}$):

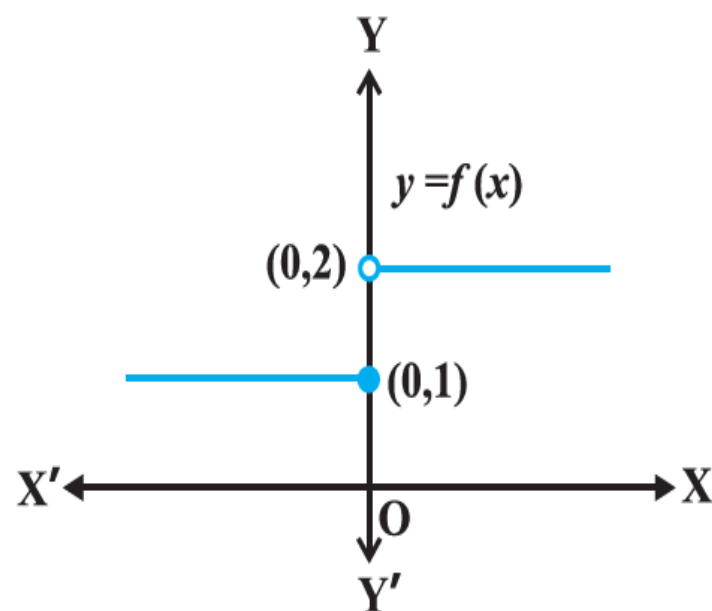
$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} [\ln(1+x)]}{\frac{d}{dx} [x]} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{1} = \lim_{x \rightarrow 0} \frac{1}{1+x} = 1$$

Continuity

We start the section with two informal examples to get a feel of continuity. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 2, & \text{if } x > 0 \end{cases}$$

This function is of course defined at every point of the real line. Graph of this function is given in the Fig



Fig

Now, consider the function defined as

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$

This function is also defined at every point. Left and the right hand limits at $x = 0$ are both equal to 1. But the value of the function at $x = 0$ equals 2 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is yet another instance of a function being not continuous at $x = 0$.

Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function *around* that point without lifting the pen from the plane of the paper.

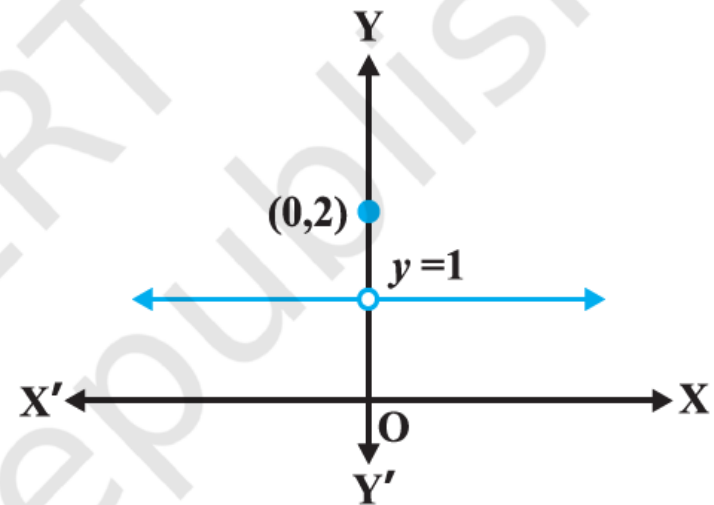


Fig 5.2

$$\lim_{x \rightarrow c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other, then f is said to be continuous at $x = c$.

if the right hand and left hand limits at $x = c$ coincide, then we say that the common value is the limit of the function at $x = c$. Hence we may also rephrase the definition of continuity as follows: *a function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.*

If f is not continuous at c , we say f is *discontinuous* at c and c is called a *point of discontinuity* of f .

उदाहरण 2. $x = 2$ पर फलन $f(x) = 2x^2 - 3x + 1$ की सांतत्य (continuity) की जाँच कीजिए।

हल— $x = 2$ पर $f(x)$ की वामावर्ती सीमा (left handed limit)

$$f(2 - 0)$$

या

$$\begin{aligned}\lim_{x \rightarrow 2^-} f(x) &= \lim_{h \rightarrow 0} f(2 - h) \\&= \lim_{h \rightarrow 0} [2(2 - h)^2 - 3(2 - h) + 1] \\&= \lim_{h \rightarrow 0} [2(4 - 4h + h^2) - 6 + 3h + 1] \\&= \lim_{h \rightarrow 0} [3 - 5h + 2h^2] = 3 \quad \dots(1)\end{aligned}$$

$x = 2$ पर $f(x)$ की दक्षिणावर्ती सीमा (Right handed limit)

$$\begin{aligned}f(2 + 0) \text{ या } &= \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(2 + h) \\&= \lim_{h \rightarrow 0} [2(2 + h)^2 - 3(2 + h) + 1] \\&= \lim_{h \rightarrow 0} [2(4 + 4h + h^2) - 6 - 3h + 1] \\&= \lim_{h \rightarrow 0} [3 + 5h + 2h^2] \\&= 3 \quad \dots(2)\end{aligned}$$

तथा

$$\begin{aligned}f(2) &= 2(2)^2 - 3(2) + 1 \\&= 3 \quad \dots(3)\end{aligned}$$

(1), (2) व (3) से, $f(2 - 0) = f(2 + 0) = f(2)$

अतः $x = 2$ पर $f(x)$ संतत है।

उदाहरण 5. $x = 2$ पर $f(x)$ की सांतत्य (Continuity) का परीक्षण कीजिए,

$$\text{जहाँ } f(x) = \begin{cases} 1+x & \text{यदि } x \leq 2 \\ 5-x & \text{यदि } x \geq 2 \end{cases}$$

हल—दिए हुए फलन $f(x)$ की $x = 2$ वामावर्ती सीमा (L.H. Limit)

$$f(2-0)$$

या

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{h \rightarrow 0} f(2-h) \\ &= \lim_{h \rightarrow 0} [1 + (2-h)] \end{aligned}$$

[क्योंकि $x < 2$ या $x = 2-h$ पर $f(x) = 1+x$]

$$= 3$$

...(1)

दिए हुए फलन $f(x)$ की $x = 2$ पर दक्षिणावर्ती सीमा (R.H. Limit)

$$f(2+0)$$

या

$$\begin{aligned} \lim_{h \rightarrow 2^+} f(x) &= \lim_{h \rightarrow 0} f(2+h) \\ &= \lim_{h \rightarrow 0} [5 - (2+h)] \end{aligned}$$

[क्योंकि $x > 2$ या $x = 2+h$ पर $f(x) = 5-x$]

$$= \lim_{h \rightarrow 0} (3-h) = 3$$

...(2)

और

$$f(2) = 1 + 2 = 3$$

...(3)

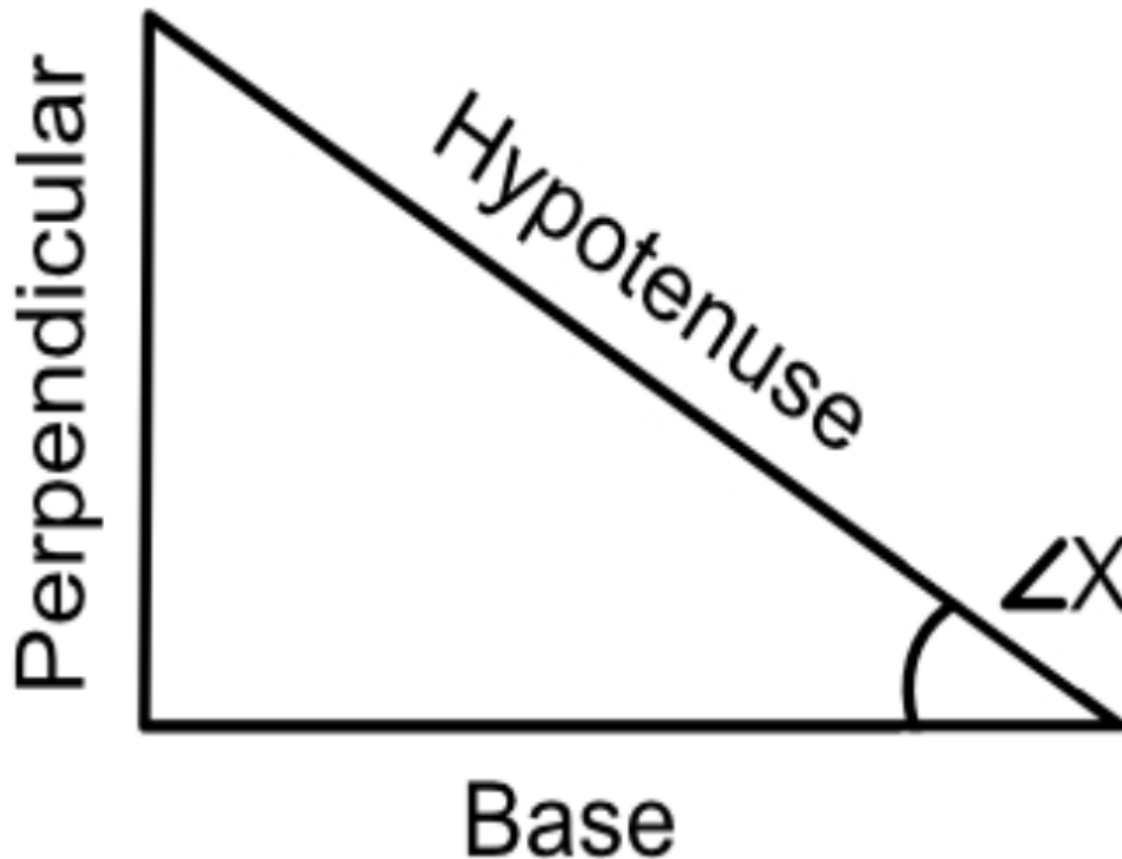
(1), (2) व (3) से,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

अतः $x = 2$ पर दिया हुआ फलन संतत (Continuous) है।

Trigonometric Formulas

Before getting into the trigonometric formula list, let us consider the following right-angled triangle:



- a. **Base:** The side that is horizontal to the plane.
- b. **Perpendicular:** The side making an angle of 90 degree with the Base.
- c. **Hypotenuse:** The longest side of the triangle.

Also, θ is the angle made by Hypotenuse and Base.

Then,

$$\text{sine of angle } \theta = \sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$$

$$\text{cosine of angle } \theta = \cos \theta = \frac{\text{Base}}{\text{Hypotenuse}}$$

$$\text{tangent of angle } \theta = \tan \theta = \frac{\text{Perpendicular}}{\text{Base}}$$

$$\text{cotangent of angle } \theta = \cot \theta = \frac{\text{Base}}{\text{Perpendicular}}$$

$$\text{cosecant of angle } \theta = \text{cosec} \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$$

$$\text{secant of angle } \theta = \sec \theta = \frac{\text{Hypotenuse}}{\text{Base}}$$

Reciprocal Relationship Between Trigonometric Functions

The reciprocal relationship between different Trigonometric Functions are as under:

$$\tan \theta = \frac{1}{\cot \theta} = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

Trigonometric Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$$

Trigonometric Functions Of Sum And Difference Of Two Angles

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Trigonometry Formulas Involving Double Angle Identities

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Trigonometry Formulas Involving Triple Angle Identities

$$\sin 3A = 3 \sin A - 4 \sin^3 A = 4 \sin(60^\circ - A) \cdot \sin A \cdot \sin(60^\circ + A)$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A = 4 \cos(60^\circ - A) \cdot \cos A \cdot \cos(60^\circ + A)$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} = \tan(60^\circ - A) \cdot \tan A \cdot \tan(60^\circ + A)$$

Trigonometry Formulas: Inverse Properties

$\theta = \sin^{-1}(x)$ is equivalent to $x = \sin \theta$

$\theta = \cos^{-1}(x)$ is equivalent to $x = \cos \theta$

$\theta = \tan^{-1}(x)$ is equivalent to $x = \tan \theta$

$$\sin(\sin^{-1}(x)) = x$$

$$\cos(\cos^{-1}(x)) = x$$

$$\tan(\tan^{-1}(x)) = x$$

$$\sin^{-1}(\sin(\theta)) = \theta$$

$$\cos^{-1}(\cos(\theta)) = \theta$$

$$\tan^{-1}(\tan(\theta)) = \theta$$

Trigonometry Formulas Involving Product Identities

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$$

Trigonometry Formulas Involving Sum To Product Identities

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

Below is the table for trigonometry formulas for angles that are commonly used for solving problems.

Angles (In Degrees)	0°	30°	45°	60°	90°
Angles (In Radians)	0°	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0
cosec	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1
sec	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞

Differentiation

The rate of change of a quantity y with respect to another quantity x is called the **derivative** or differential coefficient of y with respect to x .

The process of finding derivative of a function is called differentiation.

Differentiation of a Function

This new function is said to be differentiation (differential coefficient) of the function $f(x)$ with respect to x and it is denoted by $\frac{df(x)}{dx}$ or $Df(x)$

or $f'(x)$.
$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Differentiation from First Principle

Let $f(x)$ is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Differentiation of standard function

$$(i) \frac{d}{dx} (x^n) = nx^{n-1}$$

$$(ii) \frac{d}{dx} (k) = 0, \text{ where } k \text{ is constant.}$$

$$(iii) \frac{d}{dx} (e^x) = e^x$$

$$(iv) \frac{d}{dx} (a^x) = a^x \log_e a, \text{ where } a > 0, a \neq 1$$

$$(v) \frac{d}{dx} (\log_e x) = \frac{1}{x}, x > 0$$

$$(xiii) \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},$$

$$(xiv) \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}},$$

$$(xv) \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$(vi) \frac{d}{dx} (\log_a x) = \frac{1}{x} (\log_a e) = \frac{1}{x \log_e a}$$

$$(vii) \frac{d}{dx} (\sin x) = \cos x$$

$$(viii) \frac{d}{dx} (\cos x) = -\sin x$$

$$(ix) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$(x) \frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x,$$

$$(xi) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$(xii) \frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x,$$

$$(xvi) \frac{d}{dx} (\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$(xvii) \frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2 - 1}}$$

$$(xviii) \frac{d}{dx} (\operatorname{cosec}^{-1} x) = - \frac{1}{|x| \sqrt{x^2 - 1}}$$

$$(xix) \frac{d}{dx} (\sinh x) = \cosh x$$

$$(xx) \frac{d}{dx} (\cosh x) = \sinh x$$

$$(xxi) \frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$$

$$(xxii) \frac{d}{dx} (\coth x) = - \operatorname{cosech}^2 x$$

$$(xxiii) \frac{d}{dx} (\operatorname{sech} x) = - \operatorname{sech} x \tanh x$$

$$(xxiv) \frac{d}{dx} (\operatorname{cosech} x) = - \operatorname{cosech} x \coth x$$

$$(xxv) \frac{d}{dx} (\sinh^{-1} x) = 1 / \sqrt{x^2 + 1}$$

$$(xxvi) \frac{d}{dx} (\cosh^{-1} x) = 1 / \sqrt{x^2 - 1}$$

$$(xxvii) \frac{d}{dx} (\tanh^{-1} x) = 1 / (1 - x^2)$$

$$(xxviii) \frac{d}{dx} (\coth^{-1} x) = 1 / (1 - x^2)$$

$$(xxix) \frac{d}{dx} (\operatorname{sech}^{-1} x) = - 1 / x \sqrt{1 - x^2}$$

$$(xxx) \frac{d}{dx} (\operatorname{cosech}^{-1} x) = - 1 / x \sqrt{1 + x^2}$$

Differentiating functions using the **first principle** involves finding the derivative by applying the definition of a derivative. The definition is:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

1. $f(x) = \sin(x)$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Using the trigonometric identity:

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h),$$

we get:

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}.$$

Factoring terms:

$$f'(x) = \lim_{h \rightarrow 0} \left[\sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h} \right].$$

As $h \rightarrow 0$: $\frac{\cos(h)-1}{h} \rightarrow 0$, $\frac{\sin(h)}{h} \rightarrow 1$.

Thus:

$$f'(x) = \sin(x)(0) + \cos(x)(1) = \cos(x).$$

$$2. f(x) = \cos(x)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h}.$$

Using the trigonometric identity:

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h),$$

we get:

$$f'(x) = \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}.$$

Factoring terms:

$$f'(x) = \lim_{h \rightarrow 0} \left[\cos(x) \frac{\cos(h) - 1}{h} - \sin(x) \frac{\sin(h)}{h} \right].$$

As $h \rightarrow 0$:

$$\frac{\cos(h)-1}{h} \rightarrow 0, \quad \frac{\sin(h)}{h} \rightarrow 1.$$

Thus:

$$f'(x) = \cos(x)(0) - \sin(x)(1) = -\sin(x).$$

3. $f(x) = e^x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}.$$

Simplify:

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^x \cdot e^h - e^x}{h}.$$

Factor e^x :

$$f'(x) = e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h}.$$

Using the property:

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1,$$

we get:

$$f'(x) = e^x \cdot 1 = e^x.$$

4. Third-degree polynomial:

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Substitute $f(x+h) = a(x+h)^3 + b(x+h)^2 + c(x+h) + d$:

$$f'(x) = \lim_{h \rightarrow 0} \frac{a((x+h)^3 - x^3) + b((x+h)^2 - x^2) + c(h)}{h}$$

Expand: $(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$

$$(x+h)^2 - x^2 = 2xh + h^2$$

Substitute these: $f'(x) = \lim_{h \rightarrow 0} \frac{a(3x^2h + 3xh^2 + h^3) + b(2xh + h^2) + c(h)}{h}$

Factor h : $f'(x) = \lim_{h \rightarrow 0} [3ax^2 + 3axh + ah^2 + 2bx + bh + c]$

As $h \rightarrow 0$: $f'(x) = 3ax^2 + 2bx + c.$

If $y = ax^n$, where a and n are constants

$$\frac{dy}{dx} = n \cdot a \cdot x^{n-1}$$

$$\text{i) } y = 4x \Rightarrow \frac{dy}{dx} = 4 x^0 = 4$$

$$\text{ii) } y = 4x^2 \Rightarrow \frac{dy}{dx} = 8x$$

$$\text{iii) } y = 4x^{-2} \Rightarrow \frac{dy}{dx} = -8x^{-3}$$

Fundamental Rules for Derivatives

$$(i) \frac{d}{dx} \{cf(x)\} = c \frac{d}{dx} f(x), \text{ where } c \text{ is a constant.}$$

$$(ii) \frac{d}{dx} \{f(x) \pm g(x)\} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

$$(iii) \frac{d}{dx} \{f(x) \cdot g(x)\} = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$$

$$(iv) \frac{d}{dx} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{\{g(x)^2\}}$$

$$(v) \text{ If } \frac{d}{dx} f(x) = \phi(x), \text{ then } \frac{d}{dx} f(ax + b) = a \phi(ax + b)$$

Basic Derivatives

1. Differentiate $y = x^2 + 3x + 5$.
2. Find $\frac{d}{dx}(5x^4 - 2x + 7)$.
3. Differentiate $y = \sqrt{x}$.
4. Find $\frac{d}{dx}(x^3 - 7x^2 + 4x - 8)$.
5. Differentiate $y = \frac{1}{x^2}$.

1. Differentiate $y = 2x^5 + 3x^3 - 4x + 7$.
2. Find $\frac{d}{dx}(x^{10})$.
3. Differentiate $y = 3x^4 - 2x^3 + x^2 - 5$.
4. Compute $\frac{d}{dx}(x^{-3})$.
5. Differentiate $y = \frac{1}{x^5}$.
6. Differentiate $y = 7x + 10$.
7. Find $\frac{d}{dx}(3)$.
8. Compute $\frac{d}{dx}(5x - 2)$.
9. Differentiate $y = 2x - 9$.
10. Find $\frac{d}{dx}(x - 1)$.

11. Differentiate $y = \sqrt{x}$.
12. Compute $\frac{d}{dx}(\sqrt{3x+1})$.
13. Find $\frac{d}{dx}(x^{1/2})$.
14. Differentiate $y = \sqrt{x^2+1}$.
15. Compute $\frac{d}{dx}(\sqrt{x^3})$.
16. Differentiate $y = x^{-2}$.
17. Compute $\frac{d}{dx}(x^{-5})$.
18. Find $\frac{d}{dx}(x^{-1} + x^{-2})$.
19. Differentiate $y = 3x^{-4} + 2x^{-3}$.
20. Compute $\frac{d}{dx}(-x^{-3})$.
21. Differentiate $y = x^2 + x + 1$.
22. Compute $\frac{d}{dx}(x^3 - 3x^2 + 4x - 5)$.
23. Find $\frac{d}{dx}(x^6 - 2x^4 + 3x^2)$.
24. Differentiate $y = 5x^5 + 4x^4 - 2x^3$.
25. Compute $\frac{d}{dx}(2x^7 + 3x^5 - 6x^2)$.

26. Differentiate $y = \frac{1}{x}$.
27. Compute $\frac{d}{dx} \left(\frac{2}{x^3} \right)$.
28. Find $\frac{d}{dx} \left(\frac{1}{x^2} + \frac{2}{x} \right)$.
29. Differentiate $y = \frac{3}{x^4} + \frac{4}{x^5}$.
30. Compute $\frac{d}{dx} \left(\frac{7}{x} \right)$.
31. Differentiate $y = e^x$.
32. Compute $\frac{d}{dx}(e^{2x})$.
33. Find $\frac{d}{dx}(e^{-x})$.
34. Differentiate $y = 5e^x$.
35. Compute $\frac{d}{dx}(2e^{3x})$.
36. Differentiate $y = \ln x$.
37. Compute $\frac{d}{dx}(\ln(2x))$.
38. Find $\frac{d}{dx}(\ln(x^2 + 1))$.
39. Differentiate $y = \ln(\sqrt{x})$.
40. Compute $\frac{d}{dx}(\ln(3x + 4))$.

41. Differentiate $y = x^2 + \ln x$.

42. Compute $\frac{d}{dx}(e^x + x^3)$.

43. Find $\frac{d}{dx}(x + \ln(x^2))$.

44. Differentiate $y = e^x + \sqrt{x}$.

45. Compute $\frac{d}{dx}(x^2 + e^{-x})$.

Find the second derivative of $y = x^2$.

Compute the second derivative of $y = x^3 + 2x^2$.

Differentiate $y = x^4$ twice.

Find $\frac{d^2y}{dx^2}$ for $y = \ln x$.

Compute $\frac{d^2y}{dx^2}$ for $y = e^x$.

1. $\sin (x^2 + 5)$.

Sol. Let $y = \sin (x^2 + 5)$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \sin (x^2 + 5) = \cos (x^2 + 5) \frac{d}{dx} (x^2 + 5)$$

$$\left[\because \frac{d}{dx} \sin f(x) = \cos f(x) \frac{d}{dx} f(x) \right]$$

$$= \cos (x^2 + 5) \cdot (2x + 0)$$

$$\therefore \left[\frac{d}{dx} x^n = n x^{n-1} \text{ and } \frac{d}{dx} (c) = 0 \right]$$

$$= 2x \cos (x^2 + 5).$$

2. $\cos (\sin x)$.

Sol. Let $y = \cos (\sin x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \cos (\sin x) = -\sin (\sin x) \frac{d}{dx} \sin x$$

$$\left[\because \frac{d}{dx} \cos f(x) = -\sin f(x) \frac{d}{dx} f(x) \right]$$

$$= -\sin (\sin x) \cdot \cos x = -\cos x \sin (\sin x).$$

3. $\sin(ax + b)$.

Sol. Let $y = \sin(ax + b)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \sin(ax + b) = \cos(ax + b) \frac{d}{dx} (ax + b)$$

$$= \cos(ax + b) \left[a \frac{d}{dx}(x) + \frac{d}{dx}(b) \right]$$

$$= \cos(ax + b) [a(1) + 0]$$

$$= a \cos(ax + b).$$

4. $\sec (\tan \sqrt{x})$.

Sol. Let $y = \sec (\tan \sqrt{x})$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \sec (\tan \sqrt{x})$$

$$= \sec (\tan \sqrt{x}) \tan (\tan \sqrt{x}) \frac{d}{dx} (\tan \sqrt{x})$$

$$\left[\because \frac{d}{dx} \sec f(x) = \sec f(x) \tan f(x) \frac{d}{dx} f(x) \right]$$

$$= \sec (\tan \sqrt{x}) \tan (\tan \sqrt{x}) \sec^2 (\sqrt{x}) \frac{d}{dx} \sqrt{x}$$

$$\left[\because \frac{d}{dx} f(x) = \sec^2 f(x) \frac{d}{dx} f(x) \right]$$

$$= \sec (\tan \sqrt{x}) \tan (\tan \sqrt{x}) \sec^2 \sqrt{x} \frac{1}{2\sqrt{x}}$$

$$\left[\because \frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \right]$$

$$5. \frac{\sin(ax + b)}{\cos(cx + d)}$$

Sol. Let $y = \frac{\sin(ax + b)}{\cos(cx + d)}$

$$\therefore \frac{dy}{dx} = \frac{\cos(cx + d) \frac{d}{dx} \sin(ax + b) - \sin(ax + b) \frac{d}{dx} \cos(cx + d)}{\cos^2(cx + d)}$$

$$\left[\because \text{By Quotient Rule } \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{(\text{DEN.}) \frac{d}{dx} (\text{NUM}) - \text{NUM} \frac{d}{dx} (\text{DEN})}{(\text{DEN})^2} \right]$$

$$= \frac{\cos(cx + d) \cos(ax + b) \frac{d}{dx} (ax + b) - \sin(ax + b) (-\sin(cx + d)) \frac{d}{dx} (cx + d)}{\cos^2(cx + d)}$$

$$= \frac{a \cos(cx + d) \cos(ax + b) + c \sin(ax + b) \sin(cx + d)}{\cos^2(cx + d)}$$

$$\left[\because \frac{d}{dx} (ax + b) = \frac{d}{dx} (ax) + \frac{d}{dx} (b) = a \frac{d}{dx} (x) + 0 = a \cdot 1 = a \right]$$

$$\left[\text{Similarly } \frac{d}{dx} (cx + d) = c \right]$$

6. $\cos x^3 \sin^2 (x^5)$.

Sol. Let $y = \cos x^3 \sin^2 (x^5) = \cos x^3 (\sin x^5)^2$

$$\therefore \frac{dy}{dx} = \cos x^3 \frac{d}{dx} (\sin x^5)^2 + (\sin x^5)^2 \frac{d}{dx} \cos x^3$$

$$\left[\because \text{By Product Rule } \frac{d}{dx} (uv) = I \frac{d}{dx} (II) + II \frac{d}{dx} (I) \right]$$

$$= \cos x^3 \cdot 2 (\sin x^5) \frac{d}{dx} \sin x^5 + (\sin x^5)^2 (-\sin x^3) \frac{d}{dx} x^3$$

$$= \cos x^3 \cdot 2 (\sin x^5) \cos x^5 (5x^4) + \sin^2 x^5 (-\sin x^3) 3x^2$$

$$\left[\because \frac{d}{dx} \sin x^5 = \cos x^5 \frac{d}{dx} x^5 = \cos x^5 (5x^4) \right]$$

$$= 10x^4 \cos x^3 \sin x^5 \cos x^5 - 3x^2 \sin^2 x^5 \sin x^3$$

$$= x^2 \sin x^5 [10x^2 \cos x^3 \cos x^5 - 3 \sin x^5 \sin x^3].$$

7. $2\sqrt{\cot(x^2)}$.

Sol. Let $y = 2\sqrt{\cot(x^2)} = 2(\cot(x^2))^{1/2}$

$$\therefore \frac{dy}{dx} = 2 \cdot \frac{1}{2} (\cot x^2)^{1/2-1} \frac{d}{dx} (\cot(x^2))$$

$$\left| \because \frac{d}{dx} (f(x))^n = n(f(x))^{n-1} \frac{d}{dx} f(x) \right.$$

$$= (\cot x^2)^{-1/2} \left(-\operatorname{cosec}^2(x^2) \frac{d}{dx} x^2 \right)$$

$$\left| \because \frac{d}{dx} \cot f(x) = -\operatorname{cosec}^2(f(x)) \frac{d}{dx} f(x) \right.$$

$$= \frac{-\operatorname{cosec}^2(x^2)}{\sqrt{\cot x^2}} (2x) = \frac{-2x \operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}.$$

8. $\cos(\sqrt{x})$.

Sol. Let $y = \cos(\sqrt{x})$

$$\therefore \frac{dy}{dx} = \frac{d}{dx} \cos(\sqrt{x}) = -\sin \sqrt{x} \frac{d}{dx} \sqrt{x}$$

$$\left[\because \frac{d}{dx} \cos f(x) = -\sin f(x) \frac{d}{dx} f(x) \right]$$

$$= -\sin \sqrt{x} \frac{1}{2\sqrt{x}} \left[\because \frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \right]$$

2. $2x + 3y = \sin y$.

Sol. Given: $2x + 3y = \sin y$

Differentiating both sides w.r.t. x , we have

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx} \sin y \quad \therefore 2 + 3 \frac{dy}{dx} = \cos y \frac{dy}{dx}$$

$$\Rightarrow -\cos y \frac{dy}{dx} + 3 \frac{dy}{dx} = -2 \quad \Rightarrow -\frac{dy}{dx} (\cos y - 3) = -2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

3. $ax + by^2 = \cos y$.

Sol. Given: $ax + by^2 = \cos y$

Differentiating both sides w.r.t. x , we have

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y) \quad \therefore a + b \cdot 2y \frac{dy}{dx} = -\sin y \frac{dy}{dx}$$

$$\Rightarrow 2by \frac{dy}{dx} + \sin y \frac{dy}{dx} = -a$$

$$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a \quad \Rightarrow \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

4. $xy + y^2 = \tan x + y$.

Sol. Given: $xy + y^2 = \tan x + y$

Differentiating both sides w.r.t. x , we have

$$\frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}\tan x + \frac{d}{dx}y$$

Applying product rule,

$$x \frac{d}{dx}y + y \frac{d}{dx}x + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow x \frac{dy}{dx} + 2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x - y$$

$$\Rightarrow (x + 2y - 1) \frac{dy}{dx} = \sec^2 x - y \quad \therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

5. $x^2 + xy + y^2 = 100$.

Sol. Given: $x^2 + xy + y^2 = 100$

Differentiating both sides w.r.t. x ,

$$\frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^2 = \frac{d}{dx}(100)$$

$$\therefore 2x + \left(x \frac{d}{dx}y + y \frac{d}{dx}x \right) + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow (x + 2y) \frac{dy}{dx} = -2x - y \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{(2x + y)}{x + 2y}.$$

7. $\sin^2 y + \cos xy = \pi$.

Sol. Given: $\sin^2 y + \cos xy = \pi$

Differentiating both sides w.r.t. x ,

$$\frac{d}{dx} (\sin y)^2 + \frac{d}{dx} \cos xy = \frac{d}{dx} (\pi)$$

$$\therefore 2 (\sin y)^1 \frac{d}{dx} \sin y - \sin xy \frac{d}{dx} (xy) = 0$$

$$\Rightarrow 2 \sin y \cos y \frac{dy}{dx} - \sin xy \left(x \frac{dy}{dx} + y \cdot 1 \right) = 0$$

$$\Rightarrow \sin 2y \frac{dy}{dx} - x \sin xy \frac{dy}{dx} - y \sin xy = 0$$

$$\Rightarrow (\sin 2y - x \sin xy) \frac{dy}{dx} = y \sin xy$$

$$\therefore \frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}.$$

8. $\sin^2 x + \cos^2 y = 1.$

Sol. Given: $\sin^2 x + \cos^2 y = 1$

Differentiating both sides w.r.t. x ,

$$\frac{d}{dx} (\sin x)^2 + \frac{d}{dx} (\cos y)^2 = \frac{d}{dx} (1) \quad (1)$$

$$\therefore 2 (\sin x)^1 \frac{d}{dx} \sin x + 2 (\cos y)^1 \frac{d}{dx} \cos y = 0$$

$$\Rightarrow 2 \sin x \cos x + 2 \cos y \left(-\sin y \frac{dy}{dx} \right) = 0$$

$$\Rightarrow 2 \sin x \cos x - 2 \sin y \cos y \frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0$$

$$\Rightarrow -\sin 2y \frac{dy}{dx} = -\sin 2x \Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}.$$

$$9. \ y = \sin^{-1} \left(\frac{2x}{1+x^2} \right).$$

$$\text{Sol. Given: } y = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

To simplify the given Inverse T-function, put $x = \tan \theta$.

$$\therefore y = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\sin 2\theta) = 2\theta$$

$$\Rightarrow y = 2 \tan^{-1} x \quad (\because x = \tan \theta \Rightarrow \theta = \tan^{-1} x)$$

$$\therefore \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

10. $y = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right), \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}.$

Sol. Given: $y = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right), \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

To simplify the given Inverse T-function, put $x = \tan \theta$.

$$\therefore y = \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) = \tan^{-1} (\tan 3\theta) = 3\theta$$

$$\Rightarrow y = 3 \tan^{-1} x \quad (\because x = \tan \theta \Rightarrow \theta = \tan^{-1} x)$$

$$\therefore \frac{dy}{dx} = 3 \cdot \frac{1}{1 + x^2} = \frac{3}{1 + x^2}.$$

$$11. \ y = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), \ 0 < x < 1.$$

Sol. Given: $y = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), \ 0 < x < 1$

To simplify the given Inverse T-function, put $x = \tan \theta$.

$$\begin{aligned} \therefore \quad y &= \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) = \cos^{-1} (\cos 2\theta) \\ &= 2\theta = 2 \tan^{-1} x \quad (\because x = \tan \theta \Rightarrow \theta = \tan^{-1} x) \end{aligned}$$

$$\therefore \quad \frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

$$12. \ y = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right), \ 0 < x < 1.$$

Sol. Given: $y = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$

To simplify the given Inverse T-function, **put $x = \tan \theta$.**

$$\therefore y = \sin^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\cos 2\theta)$$

$$= \sin^{-1} \sin \left(\frac{\pi}{2} - 2\theta \right) = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow y = \frac{\pi}{2} - 2 \tan^{-1} x \quad (\because x = \tan \theta \Rightarrow \theta = \tan^{-1} x)$$

$$\therefore \frac{dy}{dx} = 0 - 2 \cdot \frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

$$13. \ y = \cos^{-1} \left(\frac{2x}{1+x^2} \right), \ -1 < x < 1.$$

Sol. Given: $y = \cos^{-1} \left(\frac{2x}{1+x^2} \right)$

To simplify the given Inverse T-function **put** $x = \tan \theta$.

$$\therefore \quad y = \cos^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \cos^{-1} (\sin 2\theta)$$

$$= \cos^{-1} \cos \left(\frac{\pi}{2} - 2\theta \right) = \frac{\pi}{2} - 2\theta$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2 \tan^{-1} x \quad (\because x = \tan \theta \Rightarrow \theta = \tan^{-1} x)$$

$$\therefore \quad \frac{dy}{dx} = 0 - 2 \cdot \frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

15. $y = \sec^{-1} \left(\frac{1}{2x^2 - 1} \right) \quad 0 < x < \frac{1}{\sqrt{2}}.$

Sol. Given: $y = \sec^{-1} \left(\frac{1}{2x^2 - 1} \right)$

To simplify the given inverse T-function, **put $x = \cos \theta$.**

$$\begin{aligned} \therefore y &= \sec^{-1} \left(\frac{1}{2 \cos^2 \theta - 1} \right) = \sec^{-1} \left(\frac{1}{\cos 2\theta} \right) \\ &= \sec^{-1} (\sec 2\theta) = 2\theta = 2 \cos^{-1} x \quad (\because x = \cos \theta \Rightarrow \theta = \cos^{-1} x) \end{aligned}$$

$$\frac{dy}{dx} = 2 \left(\frac{-1}{\sqrt{1-x^2}} \right) = \frac{-2}{\sqrt{1-x^2}}.$$

1. $\frac{e^x}{\sin x}$.

Sol. Let $y = \frac{e^x}{\sin x}$

$$\therefore \frac{dy}{dx} = \frac{(\text{DEN}) \frac{d}{dx} (\text{NUM}) - (\text{NUM}) \frac{d}{dx} (\text{DEN})}{(\text{DEN})^2}$$

$$= \frac{\sin x \frac{d}{dx} e^x - e^x \frac{d}{dx} \sin x}{\sin^2 x} = \frac{\sin x \cdot e^x - e^x \cos x}{\sin^2 x}$$

$$= e^x \frac{(\sin x - \cos x)}{\sin^2 x}.$$

2. $e^{\sin^{-1} x}$.

Sol. Let $y = e^{\sin^{-1} x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= e^{\sin^{-1} x} \frac{d}{dx} \sin^{-1} x & \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x) \right] \\ &= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}.\end{aligned}$$

3. e^{x^3} .

Sol. Let $y = e^{x^3} = e^{(x^3)}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= e^{(x^3)} \frac{d}{dx} x^3 & \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x) \right] \\ &= e^{(x^3)} 3x^2 = 3x^2 e^{(x^3)}.\end{aligned}$$

4. $\sin (\tan^{-1} e^{-x})$.

Sol. Let $y = \sin (\tan^{-1} e^{-x})$

$$\therefore \frac{dy}{dx} = \cos (\tan^{-1} e^{-x}) \frac{d}{dx} (\tan^{-1} e^{-x})$$

$$\left[\because \frac{d}{dx} \sin f(x) = \cos f(x) \frac{d}{dx} f(x) \right]$$

$$= \cos (\tan^{-1} e^{-x}) \frac{1}{1 + (e^{-x})^2} \frac{d}{dx} e^{-x}$$

$$\left[\because \frac{d}{dx} \tan^{-1} f(x) = \frac{1}{1 + (f(x))^2} \frac{d}{dx} f(x) \right]$$

$$= \cos (\tan^{-1} e^{-x}) \frac{1}{1 + e^{-2x}} e^{-x} \frac{d}{dx} (-x)$$

$$= - \frac{e^{-x} \cos (\tan^{-1} e^{-x})}{1 + e^{-2x}} \quad \left[\because \frac{d}{dx} (-x) = -1 \right]$$

5. $\log (\cos e^x)$.

Sol. Let $y = \log (\cos e^x)$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{\cos e^x} \frac{d}{dx} (\cos e^x) \left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x) \right] \\ &= \frac{1}{\cos e^x} (-\sin e^x) \frac{d}{dx} e^x \left[\because \frac{d}{dx} \cos f(x) = -\sin f(x) \frac{d}{dx} f(x) \right] \\ &= -(\tan e^x) e^x = -e^x (\tan e^x)\end{aligned}$$

6. $e^x + e^{x^2} + \dots + e^{x^5}$.

Sol. Let $y = e^x + e^{x^2} + \dots + e^{x^5}$
 $= e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx} e^x + \frac{d}{dx} e^{x^2} + \frac{d}{dx} e^{x^3} + \frac{d}{dx} e^{x^4} + \frac{d}{dx} e^{x^5} \\ &= e^x + e^{x^2} \frac{d}{dx} x^2 + e^{x^3} \frac{d}{dx} x^3 + e^{x^4} \frac{d}{dx} x^4 \\ &\quad + e^{x^5} \frac{d}{dx} x^5 \quad \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x) \right] \\ &= e^x + e^{x^2} \cdot 2x + e^{x^3} \cdot 3x^2 + e^{x^4} \cdot 4x^3 + e^{x^5} \cdot 5x^4 \\ &= e^x + 2x e^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}.\end{aligned}$$

7. $\sqrt{e^{\sqrt{x}}}$, $x > 0$.

Sol. Let $y = \sqrt{e^{\sqrt{x}}} = (e^{\sqrt{x}})^{1/2}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{1}{2} (e^{\sqrt{x}})^{-1/2} \frac{d}{dx} e^{\sqrt{x}} \left[\because \frac{d}{dx} (f(x))^n = n(f(x))^{n-1} \frac{d}{dx} f(x) \right] \\&= \frac{1}{2\sqrt{e^{\sqrt{x}}}} e^{\sqrt{x}} \frac{d}{dx} \sqrt{x} \left[\because \frac{d}{dx} e^{f(x)} = e^{f(x)} \frac{d}{dx} f(x) \right] \\&= \frac{1}{2\sqrt{e^{\sqrt{x}}}} e^{\sqrt{x}} \frac{1}{2\sqrt{x}} \left[\because \frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \right] \\&= \frac{e^{\sqrt{x}}}{4\sqrt{x} \sqrt{e^{\sqrt{x}}}}.\end{aligned}$$

9. $\frac{\cos x}{\log x}, x > 0.$

Sol. Let $y = \frac{\cos x}{\log x}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{(\text{DEN}) \frac{d}{dx} (\text{NUM}) - (\text{NUM}) \frac{d}{dx} (\text{DEN})}{(\text{DEN})^2} \\&= \frac{\log x \frac{d}{dx} (\cos x) - \cos x \frac{d}{dx} \log x}{(\log x)^2} \\&= \frac{\log x (-\sin x) - \cos x \cdot \frac{1}{x}}{(\log x)^2} \\&= \frac{-\left(\sin x \log x + \frac{\cos x}{x}\right)}{(\log x)^2} = -\frac{(x \sin x \log x + \cos x)}{x (\log x)^2}.\end{aligned}$$

1. $\cos x \cos 2x \cos 3x$.

Sol. Let $y = \cos x \cos 2x \cos 3x$... (i)

Taking logs on both sides, we have (see Note, (ii) page 261)

$$\begin{aligned}\log y &= \log (\cos x \cos 2x \cos 3x) \\ &= \log \cos x + \log \cos 2x + \log \cos 3x\end{aligned}$$

Differentiating both sides w.r.t. x , we have

$$\begin{aligned}\frac{d}{dx} \log y &= \frac{d}{dx} \log \cos x + \frac{d}{dx} \log \cos 2x + \frac{d}{dx} \log \cos 3x \\ \therefore \frac{1}{y} \frac{dy}{dx} &= \frac{1}{\cos x} \frac{d}{dx} \cos x + \frac{1}{\cos 2x} \frac{d}{dx} \cos 2x \\ &\quad + \frac{1}{\cos 3x} \frac{d}{dx} \cos 3x \left[\because \frac{d}{dx} \log f(x) = \frac{1}{f(x)} \frac{d}{dx} f(x) \right] \\ &= \frac{1}{\cos x} (-\sin x) + \frac{1}{\cos 2x} (-\sin 2x) \frac{d}{dx} (2x) \\ &\quad + \frac{1}{\cos 3x} (-\sin 3x) \frac{d}{dx} 3x \\ &= -\tan x - (\tan 2x) 2 - \tan 3x (3) \\ \therefore \frac{dy}{dx} &= -y (\tan x + 2 \tan 2x + 3 \tan 3x).\end{aligned}$$

Putting the value of y from (i),

$$\frac{dy}{dx} = -\cos x \cos 2x \cos 3x (\tan x + 2 \tan 2x + 3 \tan 3x).$$

3. $(\log x)^{\cos x}$.

Sol. Let $y = (\log x)^{\cos x}$... (i) [Form $(f(x))^{g(x)}$]

Taking logs on both sides of (i), we have (see Note (i) page 261)

$$\log y = \log (\log x)^{\cos x} = \cos x \log (\log x)$$

$[\because \log m^n = n \log m]$

$$\therefore \frac{d}{dx} \log y = \frac{d}{dx} [\cos x \cdot \log (\log x)]$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \cos x \frac{d}{dx} \log (\log x) + \log (\log x) \frac{d}{dx} \cos x$$

[By Product Rule]

$$= \cos x \frac{1}{\log x} \frac{d}{dx} \log x + \log (\log x)(-\sin x)$$

$$= \frac{\cos x}{\log x} \cdot \frac{1}{x} - \sin x \log (\log x)$$

$$\therefore \frac{dy}{dx} = y \left[\frac{\cos x}{x \log x} - \sin x \log (\log x) \right].$$

Putting the value of y from (i),

$$\frac{dy}{dx} = (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \sin x \log (\log x) \right].$$

Increasing and Decreasing Functions

A function $f(x)$ can be classified as **increasing**, **decreasing**, or **constant** over an interval based on the nature of its derivative $f'(x)$.

Definitions

1. Increasing Function

A function $f(x)$ is said to be increasing on an interval I if:

$$f'(x) > 0 \quad \forall x \in I$$

This implies that as x increases, $f(x)$ also increases.

2. Decreasing Function

A function $f(x)$ is said to be decreasing on an interval I if:

$$f'(x) < 0 \quad \forall x \in I$$

This implies that as x increases, $f(x)$ decreases.

3. Constant Function

A function $f(x)$ is constant on an interval I if:

$$f'(x) = 0 \quad \forall x \in I$$

This implies $f(x)$ remains unchanged for all x in the interval.

Steps to Determine if a Function is Increasing or Decreasing

1. Find the derivative $f'(x)$.
2. Solve $f'(x) = 0$ to find critical points where the slope changes.
3. Analyze the sign of $f'(x)$:
 - Test the intervals divided by critical points.
 - If $f'(x) > 0$, $f(x)$ is increasing.
 - If $f'(x) < 0$, $f(x)$ is decreasing.

Example

Let $f(x) = x^3 - 3x^2 + 4$.

1. Find the derivative:

$$f'(x) = 3x^2 - 6x$$

2. Solve $f'(x) = 0$:

$$3x^2 - 6x = 0 \implies x(x - 2) = 0 \implies x = 0, x = 2$$

3. Analyze the intervals $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$:

- For $x \in (-\infty, 0)$, $f'(x) = 3x^2 - 6x > 0$ (increasing).
- For $x \in (0, 2)$, $f'(x) = 3x^2 - 6x < 0$ (decreasing).
- For $x \in (2, \infty)$, $f'(x) = 3x^2 - 6x > 0$ (increasing).

4. Conclusion:

- $f(x)$ is increasing on $(-\infty, 0) \cup (2, \infty)$.
- $f(x)$ is decreasing on $(0, 2)$.

Example 1

Function: $f(x) = x^2 - 4x + 3$

1. Find the derivative:

$$f'(x) = 2x - 4$$

2. Solve $f'(x) = 0$:

$$2x - 4 = 0 \implies x = 2$$

3. Analyze the intervals $(-\infty, 2)$ and $(2, \infty)$:

- For $x \in (-\infty, 2)$: Pick $x = 0$, then $f'(0) = 2(0) - 4 = -4$ (negative, so $f(x)$ is decreasing).
- For $x \in (2, \infty)$: Pick $x = 3$, then $f'(3) = 2(3) - 4 = 2$ (positive, so $f(x)$ is increasing).

4. Conclusion:

- $f(x)$ is decreasing on $(-\infty, 2)$.
- $f(x)$ is increasing on $(2, \infty)$.

Example 2

Function: $f(x) = 3x^3 - 9x^2 + 6x$

1. Find the derivative:

$$f'(x) = 9x^2 - 18x + 6$$

2. Solve $f'(x) = 0$:

$$9x^2 - 18x + 6 = 0 \implies 3x^2 - 6x + 2 = 0 \implies (3x - 2)(x - 1) = 0$$

$$x = \frac{2}{3}, \quad x = 1$$

3. Analyze the intervals $(-\infty, \frac{2}{3})$, $(\frac{2}{3}, 1)$, and $(1, \infty)$:

- For $x \in (-\infty, \frac{2}{3})$: Pick $x = 0$, then $f'(0) = 6$ (positive, so $f(x)$ is increasing).
- For $x \in (\frac{2}{3}, 1)$: Pick $x = 0.8$, then $f'(0.8) = -0.36$ (negative, so $f(x)$ is decreasing).
- For $x \in (1, \infty)$: Pick $x = 2$, then $f'(2) = 6$ (positive, so $f(x)$ is increasing).

4. Conclusion:

- $f(x)$ is increasing on $(-\infty, \frac{2}{3}) \cup (1, \infty)$.
- $f(x)$ is decreasing on $(\frac{2}{3}, 1)$.

Application of Differentiation- Growth rate, Average Cost, and Marginal cost, Marginal Cost, Marginal Revenue.

1. Growth Rate

The growth rate of a quantity is measured by its derivative relative to time.

- If $Q(t)$ is the value of a quantity (e.g., population, revenue, or production) at time t , the growth rate is given by:

$$\text{Growth Rate} = \frac{dQ}{dt}.$$

- For a percentage growth rate:

$$\text{Percentage Growth Rate} = \frac{\frac{dQ}{dt}}{Q} \times 100.$$

This measure helps in understanding how quickly a variable changes over time.

Scenario:

Suppose the population of a town at time t (in years) is modeled by:

$$P(t) = 5000e^{0.03t}$$

Here:

- $P(t)$ is the population at time t .
- The growth rate is governed by the exponential factor **0.03**, or 3% annually.

1. Find the Growth Rate (Instantaneous):

The instantaneous growth rate is the derivative of $P(t)$ with respect to t :

$$\frac{dP}{dt} = 5000 \cdot 0.03 \cdot e^{0.03t} = 150e^{0.03t}$$

At $t = 5$ years:

$$P(5) = 5000e^{0.03 \cdot 5} = 5000e^{0.15} \approx 5000 \cdot 1.1618 \approx 5809$$

$$\left. \frac{dP}{dt} \right|_{t=5} = 150e^{0.03 \cdot 5} = 150 \cdot 1.1618 \approx 174.27$$

Thus, the instantaneous growth rate at $t = 5$ years is approximately **174.27 people per year**.

2. Find the Percentage Growth Rate:

The percentage growth rate is given by:

$$\text{Percentage Growth Rate} = \frac{\frac{dP}{dt}}{P(t)} \times 100$$

At $t = 5$:

$$\text{Percentage Growth Rate} = \frac{174.27}{5809} \times 100 \approx 3\%$$

This aligns with the exponential growth rate of 3%.

2. Average Cost

The average cost (AC) is defined as the total cost divided by the quantity produced:

$$AC = \frac{C(Q)}{Q},$$

where $C(Q)$ is the total cost for producing Q units.

Using differentiation:

- The derivative of AC can be used to identify points of minimum average cost.

3. Marginal Cost

Marginal cost (MC) represents the rate of change of the total cost with respect to the quantity produced:

$$MC = \frac{dC}{dQ}.$$

- It shows how much the cost will increase if one additional unit is produced.
- Key insights:
 - When $MC < AC$, the average cost is decreasing.
 - When $MC > AC$, the average cost is increasing.
 - The marginal cost curve typically intersects the average cost curve at its minimum point.

4. Marginal Revenue

Marginal revenue (MR) is the rate of change of total revenue with respect to the quantity sold:

$$MR = \frac{dR}{dQ},$$

where $R(Q)$ is the total revenue function.

- It indicates the additional revenue earned by selling one more unit of output.
- In a perfectly competitive market, MR is constant and equal to price.
- In imperfect markets, MR decreases as quantity increases.

5. Profit Maximization

Using differentiation, firms can determine the output level that maximizes profit (π):

- Profit function: $\pi(Q) = R(Q) - C(Q)$.
- The optimal production level occurs when:

$$\frac{d\pi}{dQ} = MR - MC = 0,$$

or equivalently, when $MR = MC$.

Example

Total Cost Function:

$$C(Q) = 100 + 5Q + 0.2Q^2$$

- Marginal Cost (MC):

$$MC = \frac{dC}{dQ} = 5 + 0.4Q.$$

Total Revenue Function:

$$R(Q) = 20Q - 0.1Q^2$$

- Marginal Revenue (MR):

$$MR = \frac{dR}{dQ} = 20 - 0.2Q.$$

To maximize profit, set $MR = MC$:

$$20 - 0.2Q = 5 + 0.4Q \quad \Rightarrow \quad Q = 25 \text{ units.}$$

Example Scenario:

- Total Cost Function: $C(Q) = 100 + 5Q + 0.2Q^2$
- Total Revenue Function: $R(Q) = 20Q - 0.1Q^2$

Here, Q represents the quantity of goods produced and sold.

1. Calculate Marginal Cost (MC):

Marginal cost is the derivative of the total cost with respect to Q :

$$MC = \frac{dC}{dQ} = 5 + 0.4Q$$

2. Calculate Average Cost (AC):

Average cost is:

$$AC = \frac{C(Q)}{Q} = \frac{100 + 5Q + 0.2Q^2}{Q}$$

Simplify:

$$AC = \frac{100}{Q} + 5 + 0.2Q$$

3. Calculate Marginal Revenue (MR):

Marginal revenue is the derivative of the total revenue with respect to Q :

$$MR = \frac{dR}{dQ} = 20 - 0.2Q$$

4. Profit Maximization:

Profit (π) is the difference between total revenue and total cost:

$$\pi(Q) = R(Q) - C(Q)$$

The condition for profit maximization is:

$$MR = MC$$

Substitute $MR = 20 - 0.2Q$ and $MC = 5 + 0.4Q$:

$$20 - 0.2Q = 5 + 0.4Q$$

Solve for Q :

$$20 - 5 = 0.4Q + 0.2Q \Rightarrow 15 = 0.6Q \Rightarrow Q = 25$$

So, the profit-maximizing quantity is $Q = 25$.

5. Find Profit at $Q = 25$:

1. Total Revenue (R):

$$R(Q) = 20Q - 0.1Q^2$$

At $Q = 25$:

$$R(25) = 20(25) - 0.1(25^2) = 500 - 62.5 = 437.5$$

2. Total Cost (C):

$$C(Q) = 100 + 5Q + 0.2Q^2$$

At $Q = 25$:

$$C(25) = 100 + 5(25) + 0.2(25^2) = 100 + 125 + 125 = 350$$

3. Profit (π):

$$\pi(Q) = R(Q) - C(Q)$$

At $Q = 25$:

$$\pi(25) = 437.5 - 350 = 87.5$$

6. Verify MC and AC at $Q = 25$:

1. Marginal Cost (MC):

$$MC = 5 + 0.4Q$$

At $Q = 25$:

$$MC = 5 + 0.4(25) = 5 + 10 = 15$$

2. Average Cost (AC):

$$AC = \frac{100}{Q} + 5 + 0.2Q$$

At $Q = 25$:

$$AC = \frac{100}{25} + 5 + 0.2(25) = 4 + 5 + 5 = 14$$

If $y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$$

If y is the sum/difference of two or more functions of x:

differentiate the 2 (or more) terms separately, then add/subtract

(i) $y = 2x^2 + 3x$ then $\frac{dy}{dx} = 4x + 3$

(ii) $y = 5x + 4$ then $\frac{dy}{dx} = 5$

$$\text{i) } y = (x+2)(ax^2+bx)$$

$$\frac{dy}{dx} = (x+2)(2ax+b) + (ax^2+bx)$$

$$\text{ii) } y = (4x^3-3x+2)(2x^2+4x)$$

$$\frac{dy}{dx} = (4x^3-3x+2)(4x+4) + (2x^2+4x)(12x^2-3)$$

$$\text{If } y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Example 1

$$y = \frac{(x + 2)}{(x + 4)}$$

$$\frac{dy}{dx} = \frac{(x + 4)(1) - (x + 2)(1)}{(x + 4)^2} = \frac{-2}{(x + 4)^2}$$

Q1 : Differentiate the following w.r.t. x :

$$\frac{e^x}{\sin x}$$

Answer :

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x} \\ &= \frac{\sin x \cdot (e^x) - e^x \cdot (\cos x)}{\sin^2 x} \\ &= \frac{e^x (\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}\end{aligned}$$

Q2 : Differentiate the following w.r.t. x :

$$e^{\sin^{-1} x}$$

Answer :

$$\text{Let } y = e^{\sin^{-1} x}$$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(e^{\sin^{-1} x} \right) \\ \Rightarrow \frac{dy}{dx} &= e^{\sin^{-1} x} \cdot \frac{d}{dx} (\sin^{-1} x) \\ &= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} \\ &= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \\ \therefore \frac{dy}{dx} &= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1, 1)\end{aligned}$$

Q3 : Differentiate the following w.r.t. x :

$$e^{x^3}$$

Answer :

Let $y = e^{x^3}$

By using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}(e^{x^3}) = e^{x^3} \cdot \frac{d}{dx}(x^3) = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

$$\sqrt{x}$$

$$\frac{d}{dx} \log(\sin x)$$

Q4 : Differentiate the following w.r.t. x :

$$\sin(\tan^{-1} e^{-x})$$

Answer :

$$\text{Let } y = \sin(\tan^{-1} e^{-x})$$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\sin(\tan^{-1} e^{-x})] \\&= \cos(\tan^{-1} e^{-x}) \cdot \frac{d}{dx} (\tan^{-1} e^{-x}) \\&= \cos(\tan^{-1} e^{-x}) \cdot \frac{1}{1+(e^{-x})^2} \cdot \frac{d}{dx} (e^{-x}) \\&= \frac{\cos(\tan^{-1} e^{-x})}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx} (-x) \\&= \frac{e^{-x} \cos(\tan^{-1} e^{-x})}{1+e^{-2x}} \times (-1) \\&= \frac{-e^{-x} \cos(\tan^{-1} e^{-x})}{1+e^{-2x}}\end{aligned}$$

Q5 : Differentiate the following w.r.t. x :

$$\log(\cos e^x)$$

Answer :

$$\text{Let } y = \log(\cos e^x)$$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\log(\cos e^x)] \\ &= \frac{1}{\cos e^x} \cdot \frac{d}{dx} (\cos e^x) \\ &= \frac{1}{\cos e^x} \cdot (-\sin e^x) \cdot \frac{d}{dx} (e^x) \\ &= \frac{-\sin e^x}{\cos e^x} \cdot e^x \\ &= -e^x \tan e^x\end{aligned}$$

8. The Inverse Function Rule

$$\text{If } x = f(y) \text{ then } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

- Examples

i) $x = 3y^2$ then

$$\frac{dx}{dy} = 6y \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{6y}$$

ii) $y = 4x^3$ then

$$\frac{dy}{dx} = 12x^2 \quad \text{so} \quad \frac{dx}{dy} = \frac{1}{12x^2}$$

1. **Derivatives of Composite Functions** (Chain Rule)

If f and g are derivative functions in their domain, then $f \circ g$ is also differentiable

$$\text{i.e. } (f \circ g)'(x) = f' \{g(x)\} g'(x)$$

More easily, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Extension of Chain Rule

If y is a function of u , u is a function of v and v is a function of x . Then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}.$$

Q8 : Differentiate the following w.r.t. x :

$$\log(\log x)$$

Answer :

$$\text{Let } y = \log(\log x)$$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [\log(\log x)] \\ &= \frac{1}{\log x} \cdot \frac{d}{dx} (\log x) \\ &= \frac{1}{\log x} \cdot \frac{1}{x}\end{aligned}$$

$$= \frac{1}{x \log x}$$

Q10 : Differentiate the following w.r.t. x :

$$\cos(\log x + e^x), x > 0$$

Answer :

$$\text{Let } y = \cos(\log x + e^x)$$

By using the chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x) \\ &= -\sin(\log x + e^x) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x) \right] \\ &= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x \right) \\ &= -\left(\frac{1}{x} + e^x \right) \sin(\log x + e^x), x > 0\end{aligned}$$

Examples

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

i) $y = (ax^2 + bx)^{1/2}$

let $v = (ax^2 + bx)$, so $y = v^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2} (ax^2 + bx)^{-\frac{1}{2}} \cdot (2ax + b)$$

ii) $y = (4x^3 + 3x - 7)^4$

let $v = (4x^3 + 3x - 7)$, so $y = v^4$

$$\frac{dy}{dx} = 4 (4x^3 + 3x - 7)^3 \cdot (12x^2 + 3)$$

Example 21 Find the derivative of the function given by $f(x) = \sin(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \circ u)(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x . Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

Alternatively, We can also directly proceed as follows:

$$\begin{aligned} y = \sin(x^2) &\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin x^2) \\ &= \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2 \end{aligned}$$

Example 22 Find the derivative of $\tan (2x + 3)$.

Solution Let $f(x) = \tan (2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \circ u)(x) = v(u(x)) = v(2x + 3) = \tan (2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and

$\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2 \sec^2 (2x + 3)$$

Example 23 Differentiate $\sin (\cos (x^2))$ with respect to x .

Solution The function $f(x) = \sin (\cos (x^2))$ is a composition $f(x) = (w \circ v \circ u)(x)$ of the three functions u , v and w , where $u(x) = x^2$, $v(t) = \cos t$ and $w(s) = \sin s$. Put

$t = u(x) = x^2$ and $s = v(t) = \cos t$. Observe that $\frac{dw}{ds} = \cos s$, $\frac{ds}{dt} = -\sin t$ and $\frac{dt}{dx} = 2x$ exist for all real x . Hence by a generalisation of chain rule, we have

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} = (\cos s) \cdot (-\sin t) \cdot (2x) = -2x \sin x^2 \cdot \cos (\cos x^2)$$

Alternatively, we can proceed as follows:

$$y = \sin (\cos x^2)$$

$$\begin{aligned} \text{Therefore } \frac{dy}{dx} &= \frac{d}{dx} \sin (\cos x^2) = \cos (\cos x^2) \frac{d}{dx} (\cos x^2) \\ &= \cos (\cos x^2) (-\sin x^2) \frac{d}{dx} (x^2) \\ &= -\sin x^2 \cos (\cos x^2) (2x) \\ &= -2x \sin x^2 \cos (\cos x^2) \end{aligned}$$

Differentiate the functions with respect to x

1. $\sin(x^2 + 5)$

2. $\cos(\sin x)$

3. $\sin(ax + b)$

4. $\sec(\tan(\sqrt{x}))$

5. $\frac{\sin(ax + b)}{\cos(cx + d)}$

6. $\cos x^3 \cdot \sin^2(x^5)$

7. $2\sqrt{\cot(x^2)}$

8. $\cos(\sqrt{x})$

Alternate method

$$\begin{aligned}\frac{d}{dx} \left[\sec(\tan \sqrt{x}) \right] &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \frac{d}{dx}(\tan \sqrt{x}) \\ &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{\sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \sec^2(\sqrt{x})}{2\sqrt{x}}\end{aligned}$$

$$\cos x^3 \cdot \sin^2(x^5)$$

Answer :

The given function is $\cos x^3 \cdot \sin^2(x^5)$.

$$\begin{aligned}\frac{d}{dx}[\cos x^3 \cdot \sin^2(x^5)] &= \sin^2(x^5) \times \frac{d}{dx}(\cos x^3) + \cos x^3 \times \frac{d}{dx}[\sin^2(x^5)] \\&= \sin^2(x^5) \times (-\sin x^3) \times \frac{d}{dx}(x^3) + \cos x^3 \times 2 \sin(x^5) \cdot \frac{d}{dx}[\sin x^5] \\&= -\sin x^3 \sin^2(x^5) \times 3x^2 + 2 \sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx}(x^5) \\&= -3x^2 \sin x^3 \cdot \sin^2(x^5) + 2 \sin x^5 \cos x^5 \cos x^3 \cdot \times 5x^4 \\&= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)\end{aligned}$$

Answer :

$$\begin{aligned}& \frac{d}{dx} \left[2\sqrt{\cot(x^2)} \right] \\&= 2 \cdot \frac{1}{2\sqrt{\cot(x^2)}} \times \frac{d}{dx} [\cot(x^2)] \\&= \sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times -\operatorname{cosec}^2(x^2) \times \frac{d}{dx}(x^2) \\&= -\sqrt{\frac{\sin(x^2)}{\cos(x^2)}} \times \frac{1}{\sin^2(x^2)} \times (2x) \\&= \frac{-2x}{\sqrt{\cos x^2} \sqrt{\sin x^2} \sin x^2} \\&= \frac{-2\sqrt{2} x}{\sqrt{2 \sin x^2 \cos x^2} \sin x^2} \\&= \frac{-2\sqrt{2} x}{\sin x^2 \sqrt{\sin 2x^2}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}[\cos(\sqrt{x})] &= -\sin(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= -\sin(\sqrt{x}) \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\ &= -\sin \sqrt{x} \times \frac{1}{2} x^{-\frac{1}{2}} \\ &= \frac{-\sin \sqrt{x}}{2\sqrt{x}}\end{aligned}$$

2. Derivatives of Inverse Trigonometric Functions

Sometimes, the given function can be deduced with the help of inverse trigonometric substitution and then to find the derivative easily.

$$(i) \ 2 \sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$$

$$(ii) \ 2 \cos^{-1} x = \cos^{-1} (2x^2 - 1) \text{ or } \cos^{-1} (1 - 2x^2)$$

$$(iii) \ 2 \tan^{-1} x = \begin{cases} \sin^{-1} \left(\frac{2x}{1+x^2} \right) \\ \tan^{-1} \left(\frac{2x}{1-x^2} \right) \\ \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \end{cases}$$

$$(iv) \ 3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$$

$$(v) \ 3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x)$$

$$(vi) \ 3 \tan^{-1} x = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right)$$

$$(vii) \ \cos^{-1} x + \sin^{-1} x = \pi/2$$

$$\text{(viii)} \quad \tan^{-1} x + \cot^{-1} x = \pi/2$$

$$\text{(ix)} \quad \sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2$$

$$\text{(x)} \quad \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2} \right]$$

$$\text{(xi)} \quad \cos^{-1} x \pm \cos^{-1} y = \cos^{-1} \left[xy \mp \sqrt{(1-x^2)(1-y^2)} \right]$$

$$\text{(xii)} \quad \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left[\frac{x \pm y}{1 \mp xy} \right]$$

Q1 : Find $\frac{dy}{dx}$:

$$2x + 3y = \sin x$$

Answer :

The given relationship is $2x + 3y = \sin x$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$$

$$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Q2 : Find $\frac{dy}{dx}$:

$$2x + 3y = \sin y$$

Answer :

The given relationship is $2x + 3y = \sin y$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y)$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos y \frac{dy}{dx} \quad [\text{By using chain rule}]$$

$$\Rightarrow 2 = (\cos y - 3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

Q3: Find $\frac{dy}{dx}$;

$$ax + by^2 = \cos y$$

Answer :

The given relationship is $ax + by^2 = \cos y$

Differentiating this relationship with respect to x , we obtain

$$\begin{aligned}\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) &= \frac{d}{dx}(\cos y) \\ \Rightarrow a + b \frac{d}{dx}(y^2) &= \frac{d}{dx}(\cos y) \quad \dots(1)\end{aligned}$$

$$\text{Using chain rule, we obtain } \frac{d}{dx}(y^2) = 2y \frac{dy}{dx} \text{ and } \frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx} \quad \dots(2)$$

From (1) and (2), we obtain

$$\begin{aligned}a + b \times 2y \frac{dy}{dx} &= -\sin y \frac{dy}{dx} \\ \Rightarrow (2by + \sin y) \frac{dy}{dx} &= -a \\ \therefore \frac{dy}{dx} &= \frac{-a}{2by + \sin y}\end{aligned}$$

Q4: Find $\frac{dy}{dx}$:

$$xy + y^2 = \tan x + y$$

Answer :

The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to x , we obtain

$$\begin{aligned}\frac{d}{dx}(xy + y^2) &= \frac{d}{dx}(\tan x + y) \\ \Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(\tan x) + \frac{dy}{dx} \\ \Rightarrow \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y \frac{dy}{dx} &= \sec^2 x + \frac{dy}{dx} \quad \text{[Using product rule and chain rule]} \\ \Rightarrow y \cdot 1 + x \cdot \frac{dy}{dx} + 2y \frac{dy}{dx} &= \sec^2 x + \frac{dy}{dx} \\ \Rightarrow (x + 2y - 1) \frac{dy}{dx} &= \sec^2 x - y \\ \therefore \frac{dy}{dx} &= \frac{\sec^2 x - y}{(x + 2y - 1)}\end{aligned}$$

Q6 : Find $\frac{dy}{dx}$:

$$x^3 + x^2y + xy^2 + y^3 = 81$$

Answer :

The given relationship is $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to x , we obtain

$$\begin{aligned}\frac{d}{dx}(x^3 + x^2y + xy^2 + y^3) &= \frac{d}{dx}(81) \\ \Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) &= 0 \\ \Rightarrow 3x^2 + \left[y \frac{d}{dx}(x^2) + x^2 \frac{dy}{dx} \right] + \left[y^2 \frac{d}{dx}(x) + x \frac{d}{dx}(y^2) \right] + 3y^2 \frac{dy}{dx} &= 0 \\ \Rightarrow 3x^2 + \left[y \cdot 2x + x^2 \frac{dy}{dx} \right] + \left[y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx} \right] + 3y^2 \frac{dy}{dx} &= 0 \\ \Rightarrow (x^2 + 2xy + 3y^2) \frac{dy}{dx} + (3x^2 + 2xy + y^2) &= 0 \\ \therefore \frac{dy}{dx} &= \frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}\end{aligned}$$

Q8 : Find $\frac{dy}{dx}$:

$$\sin^2 x + \cos^2 y = 1$$

Answer :

The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(\sin^2 x + \cos^2 y) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 x) + \frac{d}{dx}(\cos^2 y) = 0$$

$$\Rightarrow 2 \sin x \cdot \frac{d}{dx}(\sin x) + 2 \cos y \cdot \frac{d}{dx}(\cos y) = 0$$

$$\Rightarrow 2 \sin x \cos x + 2 \cos y (-\sin y) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

Q9 : Find $\frac{dy}{dx}$:

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Answer :

The given relationship is $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x , we obtain

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \quad \dots(1)$$

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$$\begin{aligned}\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) &= \frac{(1+x^2) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\ &= \frac{(1+x^2) \cdot 2 - 2x \cdot [0+2x]}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2} \quad \dots(2)\end{aligned}$$

$$\text{Also, } \sin y = \frac{2x}{1+x^2}$$

$$\begin{aligned}\Rightarrow \cos y &= \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{2x}{1+x^2}\right)^2} = \sqrt{\frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2}} \\ &= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} = \frac{1-x^2}{1+x^2} \quad \dots(3)\end{aligned}$$

From (1), (2), and (3), we obtain

$$\frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

Some Standard Substitution

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a \sin \theta$ or $a \cos \theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a \tan \theta$ or $a \cot \theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $a \operatorname{cosec} \theta$
(iv)	$\sqrt{a+x}$ and $\sqrt{a-x}$	$x = a \cos 2\theta$
(v)	$a \sin x + b \cos x$	$a = r \cos \alpha, b = r \sin \alpha$
(vi)	$\sqrt{x-\alpha}$ and $\sqrt{\beta-x}$	$x = \alpha \sin^2 \theta + \beta \cos^2 \theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos \theta)$

5. Derivative of a Function with Respect to Another Function

Let $y = f(x)$ and $z = g(x)$, then the differentiation of y with respect to z is

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{f'(x)}{g'(x)}$$

Logarithmic Differentiation Functions

- (i) If a function is the product and quotient of functions such as $y = \frac{f_1(x) f_2(x) f_3(x) \dots}{g_1(x) g_2(x) g_3(x) \dots}$, we first take logarithm and then differentiate it.
- (ii) If a function is in the form of exponent of a function over another function such as $[f(x)]^{g(x)}$, then we first take logarithm and then differentiate it.

Differentiation of Infinite Series

Sometimes, the function is given in the form of infinite series, i.e. $y = \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}$, then the process to find the derivative of such infinite series is called differentiation of infinite series.

e.g. Suppose $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}}$

Then, $y = \sqrt{\log x + y} \Rightarrow y^2 = \log x + y$

Now, differentiate it by usual method.

Successive Differentiations

If the function $y = f(x)$ be differentiated with respect to x , then the result $\frac{dy}{dx}$ or $f'(x)$, so obtained is a function of x (may be a constant).

Hence, $\frac{dy}{dx}$ can again be differentiated with respect to x .

The differential coefficient of $\frac{dy}{dx}$ with respect to x is written as

$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$ or $f''(x)$. Again, the differential coefficient of $\frac{d^2y}{dx^2}$ with respect to x is written as

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3} \text{ or } f'''(x) \dots$$

Here, $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots$ are respectively known as first, second, third, ... order differential coefficients of y with respect to x . These are alternatively denoted by $f'(x), f''(x), f'''(x), \dots$ or $y_1, y_2, y_3, \dots$, respectively.

Q1 : Find the second order derivatives of the function.

$$x^2 + 3x + 2$$

Answer :

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Derivatives of Special Types of Functions

(i) If $y = f(x)^{\{f(x)\} \dots \infty}$, then $\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)\{1 - y \log f(x)\}}$

(ii) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}}$, then $\frac{dy}{dx} = \frac{f'(x)}{2y - 1}$

(iii) If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then

$$\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)} \left[\frac{f(x) \log g(y) - g(y)}{g(y) \log f(x) - f(x)} \right]$$

Basic Formula

1. $\int x^n = x^{n+1}/n+1 + C$
2. $\int \cos x = \sin x + C$
3. $\int \sin x = -\cos x + C$
4. $\int \sec^2 x = \tan x + C$
5. $\int \operatorname{cosec}^2 x = -\cot x + C$
6. $\int \sec x \tan x = \sec x + C$
7. $\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + C$
8. $\int dx/\sqrt{1-x^2} = \sin^{-1} x + C$
9. $\int dx/\sqrt{1-x^2} = -\cos^{-1} x + C$
10. $\int dx/\sqrt{1+x^2} = \tan^{-1} x + C$
11. $\int dx/\sqrt{1+x^2} = -\cot^{-1} x + C$
12. $\int e^x = e^x + C$
13. $\int a^x = a^x / \log a + C$
14. $\int dx/x \sqrt{x^2 - 1} = \sec^{-1} x + C$
15. $\int dx/x \sqrt{x^2 - 1} = \operatorname{cosec}^{-1} x + C$
16. $\int 1/x = \log |x| + c$
17. $\int \tan x = \log |\sec x| + c$
18. $\int \cot x = \log |\sin x| + c$
19. $\int \sec x = \log |\sec x + \tan x| + c$
20. $\int \operatorname{cosec} x = \log |\operatorname{cosec} x - \cot x| + c$

Integrals of some special function s

$$1. \int dx/(x^2 - a^2) = 1/2a \log |(x - a)/(x + a)| + c$$

$$2. \int dx/(a^2 - x^2) = 1/2a \log |(a + x)/(a - x)| + c$$

$$3. \int dx/(x^2 + a^2) = 1/a \tan^{-1} x/a + c$$

$$4. \int dx/\sqrt{(x^2 - a^2)} = \log |x + \sqrt{(x^2 - a^2)}| + C$$

$$5. \int dx/\sqrt{(a^2 - x^2)} = \sin^{-1} x/a + c$$

$$6. \int dx/\sqrt{(x^2 + a^2)} = \log |x + \sqrt{(x^2 + a^2)}| + C$$

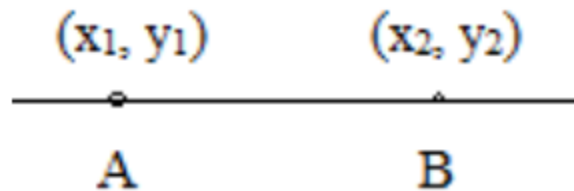
Integration by parts

$$1. \int f(x) g(x) dx = f(x) \int g(x) dx - \int (f'(x) \int g(x) dx) dx$$

Straight Lines

A straight line is a line which is not curved.

A line is simply an object in geometry that is characterized under zero width object that extends on both sides. A straight line is just a line with no curves. So, a line that extends to both sides till infinity and has no curves is called a **straight line**.



I. Distance between the points P (x_1, y_1) and Q (x_2, y_2) is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, distance between the points $(6, -4)$ and $(3, 0)$ is

$$\sqrt{(3 - 6)^2 + (0 + 4)^2} = \sqrt{9 + 16} = 5 \text{ units.}$$

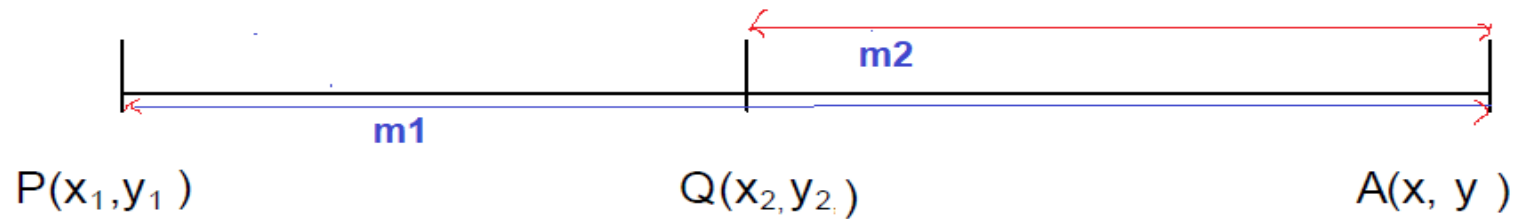
II. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally, in the ratio $m: n$ are $\left(\frac{m x_2 + n x_1}{m + n}, \frac{m y_2 + n y_1}{m + n} \right)$.

For example, the coordinates of the point which divides the line segment joining

A $(1, -3)$ and B $(-3, 9)$ internally, in the ratio $1: 3$ are given by $x = \frac{1.(-3) + 3.1}{1 + 3} = 0$

and $y = \frac{1.9 + 3.(-3)}{1 + 3} = 0$.

External section



The point A(x, y) that divides distance PQ externally in the ratio $m_1:m_2$ is given by

$$A = \left[\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2} \right]$$

III. In particular, if $m = n$, the coordinates of the mid-point of the line segment

joining the points (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

IV. Area of the triangle whose vertices are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2} \left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|.$$

For example, the area of the triangle, whose vertices are $(4, 4)$, $(3, -2)$ and $(-3, 16)$ is

$$\frac{1}{2} \left| 4(-2 - 16) + 3(16 - 4) + (-3)(4 + 2) \right| = \frac{|-54|}{2} = 27.$$

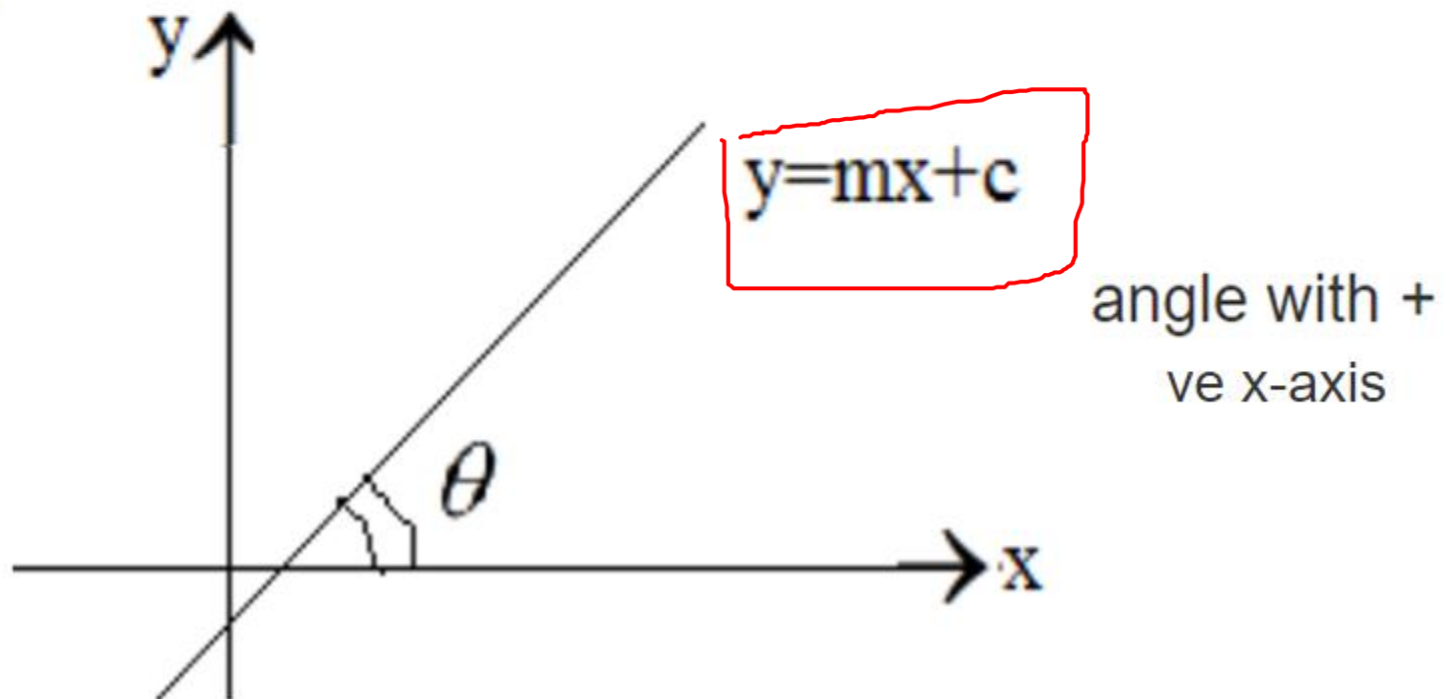
Equation of Straight Line

The general equation of straight line is as given below:

$ax + by + c = 0$ { equation of straight line. $y = -\frac{a}{b}x - \frac{c}{b}$

Where x, y are variables and a, b, c are constants.

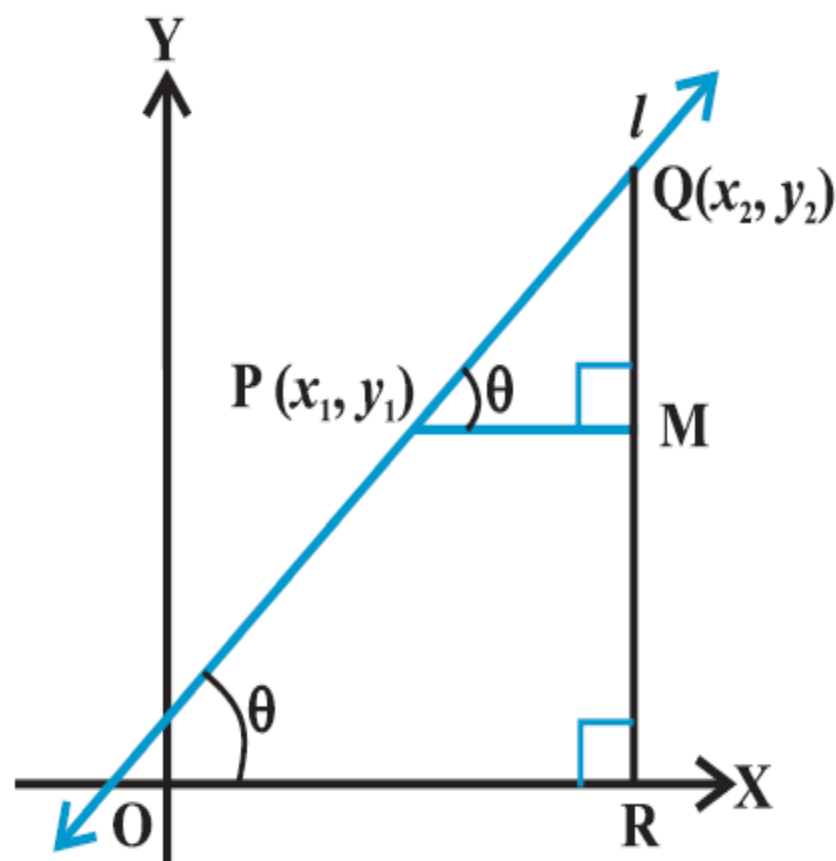
Slope:-



Slope of a line when coordinates of any two points on the line are given

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on non-vertical line l whose inclination is θ . Obviously, $x_1 \neq x_2$, otherwise the line will become perpendicular to x -axis and its slope will not be defined.

Draw perpendicular QR to x -axis and PM perpendicular to RQ as shown in Figs.



Case 1 When angle θ is acute:

In Fig 10.3 (i), $\angle MPQ = \theta$.

Therefore, slope of line $l = m = \tan \theta$ (1)

But in $\triangle MPQ$, we have $\tan \theta = \frac{MQ}{MP} = \frac{y_2 - y_1}{x_2 - x_1}$ (2)

Example Find the slope of the lines:

- (a) Passing through the points $(3, -2)$ and $(-1, 4)$,
- (b) Passing through the points $(3, -2)$ and $(7, -2)$,
- (c) Passing through the points $(3, -2)$ and $(3, 4)$,
- (d) Making inclination of 60° with the positive direction of x -axis.

Solution (a) The slope of the line through $(3, -2)$ and $(-1, 4)$ is

$$m = \frac{4 - (-2)}{-1 - 3} = \frac{6}{-4} = -\frac{3}{2}.$$

(b) The slope of the line through the points $(3, -2)$ and $(7, -2)$ is

$$m = \frac{-2 - (-2)}{7 - 3} = \frac{0}{4} = 0.$$

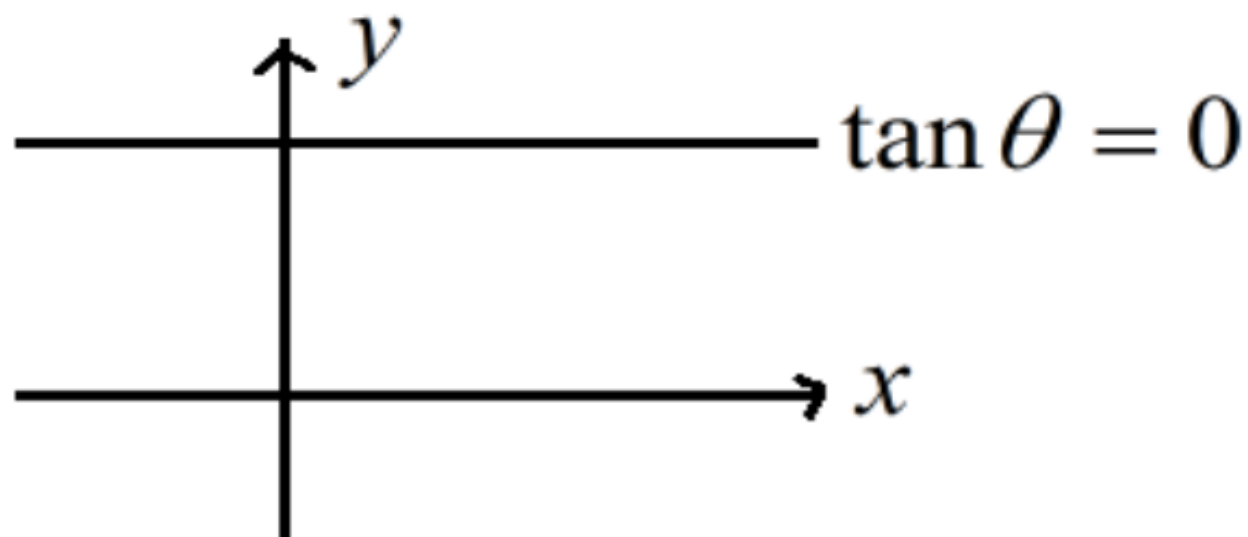
(c) The slope of the line through the points $(3, -2)$ and $(3, 4)$ is

$$m = \frac{4 - (-2)}{3 - 3} = \frac{6}{0}, \text{ which is not defined.}$$

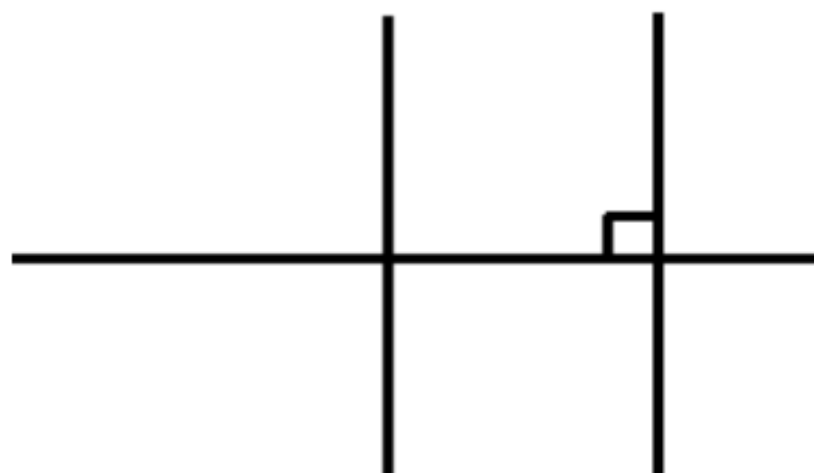
(d) Here inclination of the line $\alpha = 60^\circ$. Therefore, slope of the line is

$$m = \tan 60^\circ = \sqrt{3}.$$

Note 1 – If line is Horizontal, then slope = 0



Note 2 – If line is $\perp$ to x-axis, i.e. vertical then slope is undefined.



$$\text{Slope} = \frac{1}{0}$$

$$= \tan \frac{\pi}{2}$$

Conditions for parallelism and perpendicularity of lines in terms of their slopes

In a coordinate plane, suppose that non-vertical lines l_1 and l_2 have slopes m_1 and m_2 , respectively. Let their inclinations be α and β , respectively.

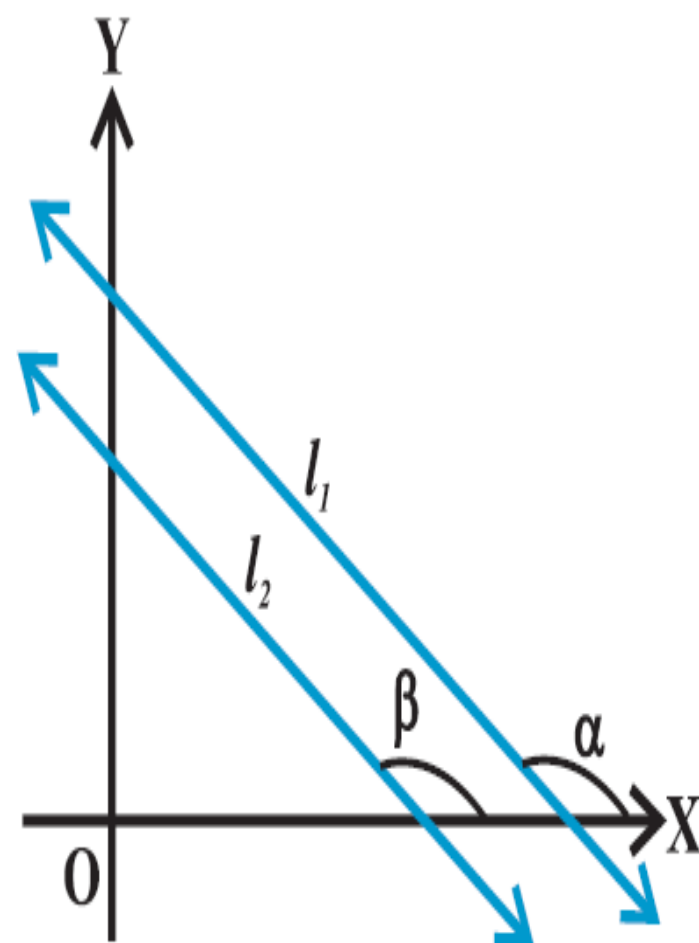
If the line l_1 is parallel to l_2 , then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan \alpha = \tan \beta$$

Therefore $m_1 = m_2$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

$$m_1 = m_2.$$



If the lines l_1 and l_2 are perpendicular, then $\beta = \alpha + 90^\circ$.

Therefore, $\tan \beta = \tan (\alpha + 90^\circ)$

$$= -\cot \alpha = -\frac{1}{\tan \alpha}$$

$$\text{i.e., } m_2 = -\frac{1}{m_1} \text{ or } m_1 m_2 = -1$$

Conversely, if $m_1 m_2 = -1$, i.e., $\tan \alpha \tan \beta = -1$.

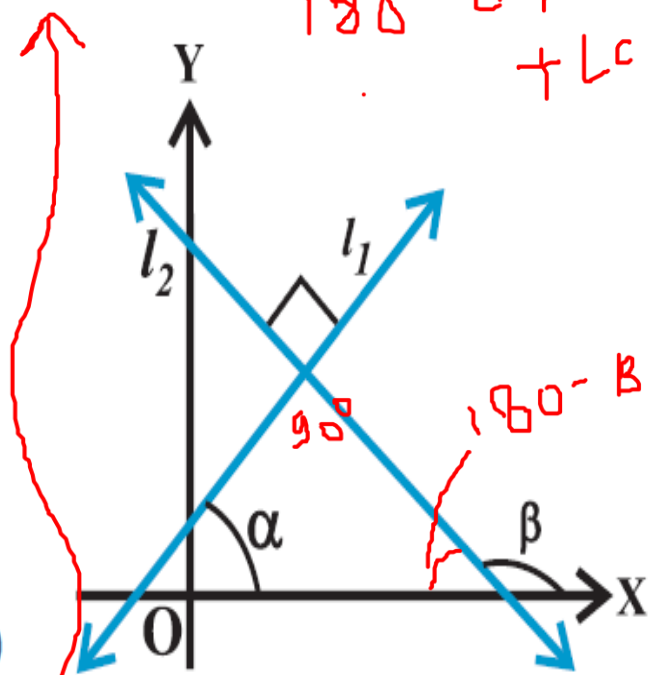
Then $\tan \alpha = -\cot \beta = \tan (\beta + 90^\circ)$ or $\tan (\beta - 90^\circ)$

Therefore, α and β differ by 90° .

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

$$\text{i.e., } m_2 = -\frac{1}{m_1} \text{ or, } m_1 m_2 = -1.$$



Fig

$$\Delta \begin{cases} \alpha + 180 = -\beta + 90 = 180 \\ \alpha + 90 = \beta \end{cases}$$

Angle between two lines

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 , respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2 , respectively. Then

$$m_1 = \tan \alpha_1 \text{ and } m_2 = \tan \alpha_2.$$

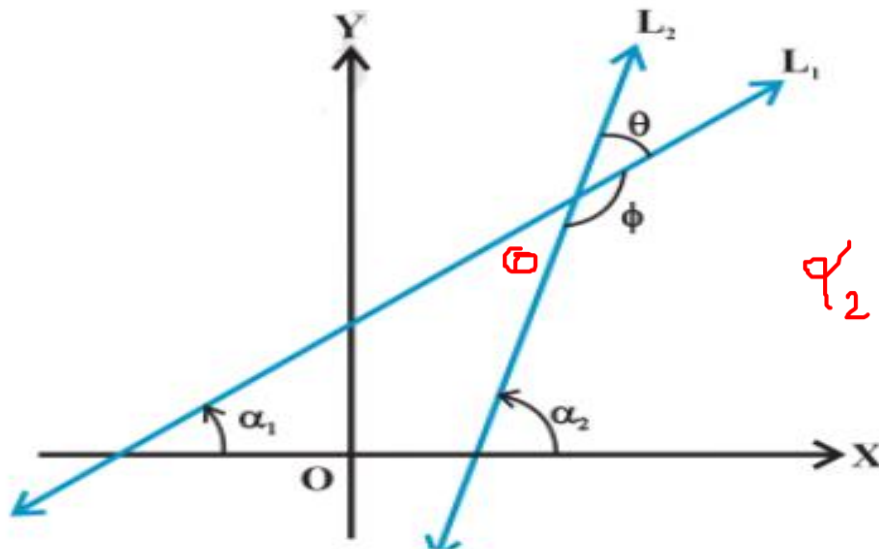
We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180° . Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Fig 10.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90^\circ.$$

$$\text{Therefore } \tan \theta = \tan (\alpha_2 - \alpha_1) = \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2} \quad (\text{as } 1 + m_1 m_2 \neq 0)$$

and $\phi = 180^\circ - \theta$ so that

$$\tan \phi = \tan (180^\circ - \theta) = -\tan \theta = -\frac{m_2 - m_1}{1 + m_1 m_2}, \text{ as } 1 + m_1 m_2 \neq 0$$



Case I If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is positive, then $\tan \theta$ will be positive

Case II If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is negative, then $\tan \theta$ will be negative

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 , respectively, is given by

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|, \text{ as } 1 + m_1 m_2 \neq 0 \quad \dots (1)$$

Example: If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution We know that the acute angle θ between two lines with slopes m_1 and m_2

is given by $\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$.

Now, putting these values in (1), we get

$$\tan \frac{\pi}{4} = \left| \frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} \right| \quad \text{or} \quad 1 = \left| \frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} \right|,$$

which gives $\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = 1$ or $\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = -1$.

Therefore $m = 3$ or $m = -\frac{1}{3}$.

Example Line through the points $(-2, 6)$ and $(4, 8)$ is perpendicular to the line through the points $(8, 12)$ and $(x, 24)$. Find the value of x .

Solution Slope of the line through the points $(-2, 6)$ and $(4, 8)$ is

$$m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Slope of the line through the points $(8, 12)$ and $(x, 24)$ is

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since two lines are perpendicular,

$m_1 m_2 = -1$, which gives

$$\frac{1}{3} \times \frac{12}{x-8} = -1 \text{ or } x = 4.$$

Horizontal and vertical lines If a horizontal line L is at a distance a from the x -axis then ordinate of every point lying on the line is either a or $-a$ [Fig 10.11 (a)]. Therefore, equation of the line L is either $y = a$ or $y = -a$. Choice of sign will depend upon the position of the line according as the line is above or below the y -axis. Similarly, the equation of a vertical line at a distance b from the y -axis is either $x = b$ or $x = -b$ [Fig 10.11(b)].

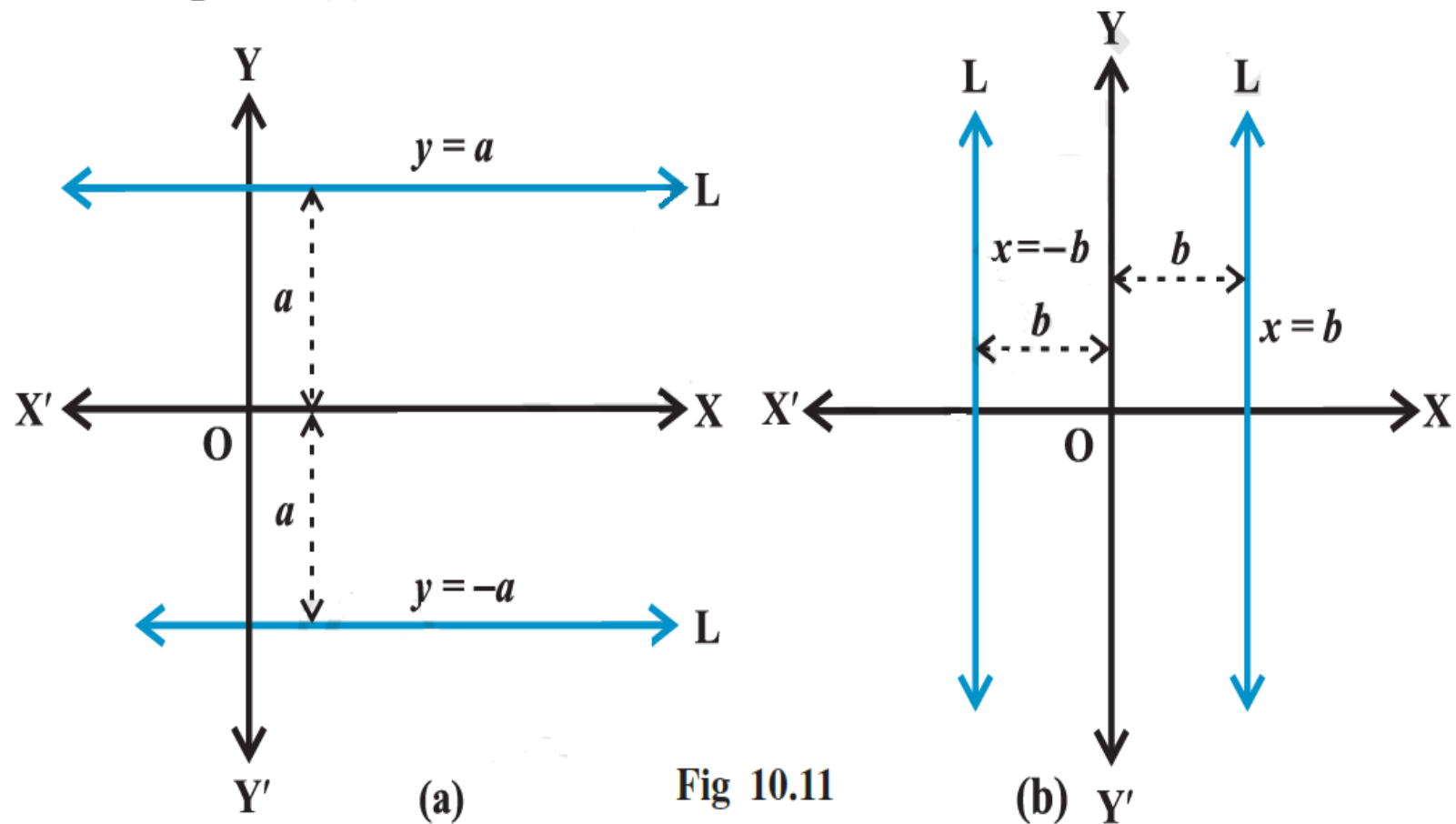


Fig 10.11

Example 6 Find the equations of the lines parallel to axes and passing through $(-2, 3)$.

Solution Position of the lines is shown in the Fig 10.12. The y -coordinate of every point on the line parallel to x -axis is 3, therefore, equation of the line parallel to x -axis and passing through $(-2, 3)$ is $y = 3$. Similarly, equation of the line parallel to y -axis and passing through $(-2, 3)$ is $x = -2$.

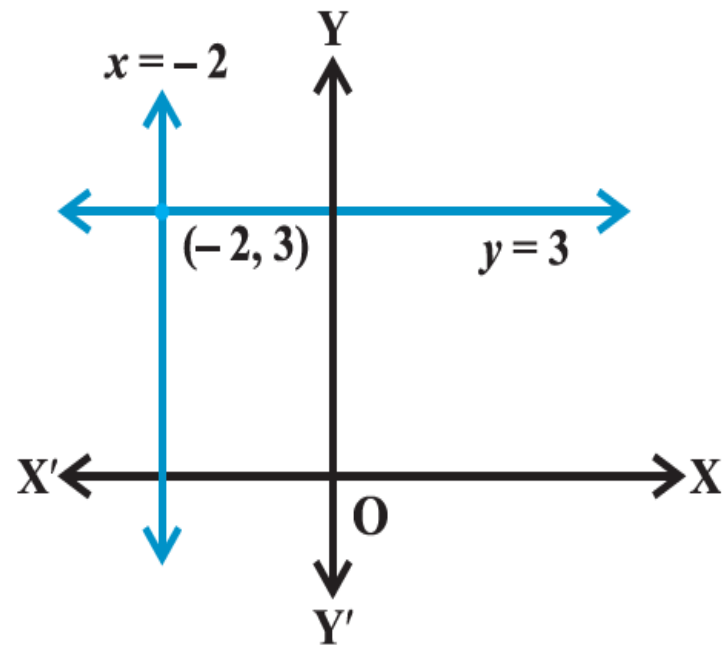


Fig 10.12

Collinearity of three points We know that slopes of two parallel lines are equal. If two lines having the same slope pass through a common point, then two lines will coincide. Hence, if A, B and C are three points in the XY-plane, then they will lie on a line, i.e., three points are collinear (Fig 10.8) if and only if slope of AB = slope of BC.

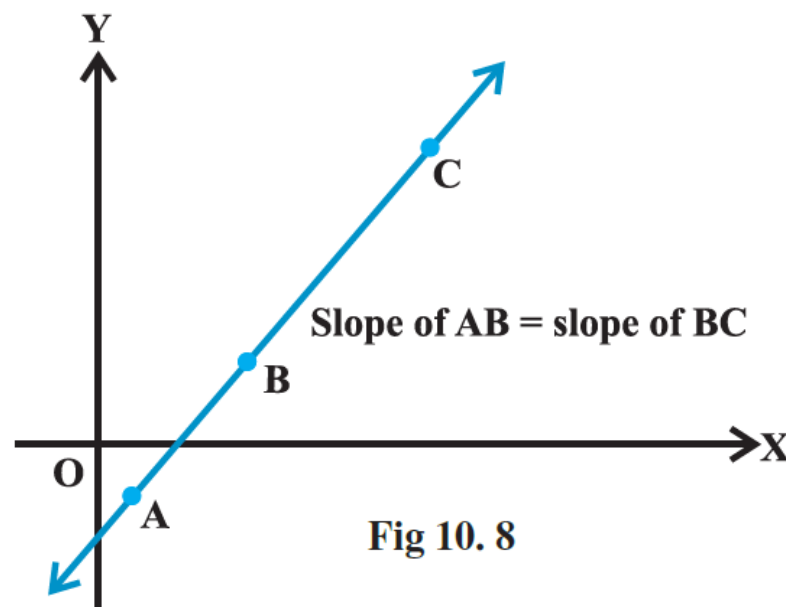


Fig 10. 8

Example 4 Three points P (h, k), Q (x_1, y_1) and R (x_2, y_2) lie on a line. Show that $(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1)$.

Solution Since points P, Q and R are collinear, we have

$$\text{Slope of PQ} = \text{Slope of QR, i.e., } \frac{y_1 - k}{x_1 - h} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{or } \frac{k - y_1}{h - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{or } (h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

Point-slope form Suppose that $P_0(x_0, y_0)$ is a fixed point on a non-vertical line L , whose slope is m . Let $P(x, y)$ be an arbitrary point on L .

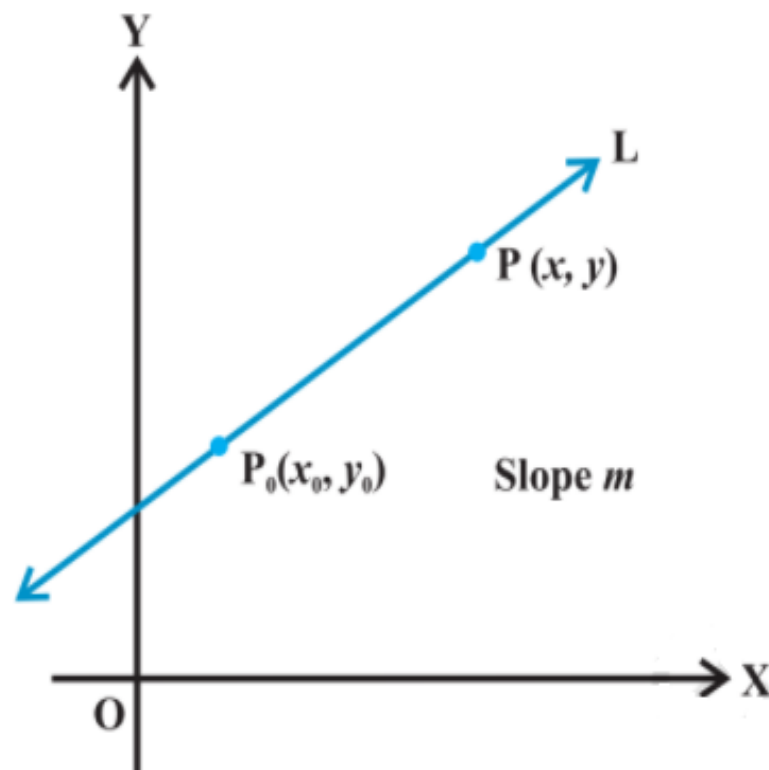
Then, by the definition, the slope of L is given by

$$m = \frac{y - y_0}{x - x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \quad \dots(1)$$

Since the point $P_0(x_0, y_0)$ along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L .

Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0) , if and only if, its coordinates satisfy the equation

$$y - y_0 = m(x - x_0)$$



Example Find the equation of the line through $(-2, 3)$ with slope -4 .

Solution Here $m = -4$ and given point (x_0, y_0) is $(-2, 3)$.

By slope-intercept form formula (1) above, equation of the given line is

$$y - 3 = -4(x + 2) \text{ or}$$

$4x + y + 5 = 0$, which is the
required equation.

Example: Find the equation of a straight line passing through $(-4,5)$

and having slope $\frac{2}{3}$

Solution: Slope = $-\frac{2}{3}$

Point $(-4,5)$

Equation of the line is $(y-y_1) = m(x-x_1)$

$$y - 5 = -\frac{2}{3}(x+4)$$

$$3y - 15 = -2x - 8$$

$\therefore$ Equation of a line is $2x + 3y - 7 = 0$

Two-point form Let the line L passes through two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $P(x, y)$ be a general point on L

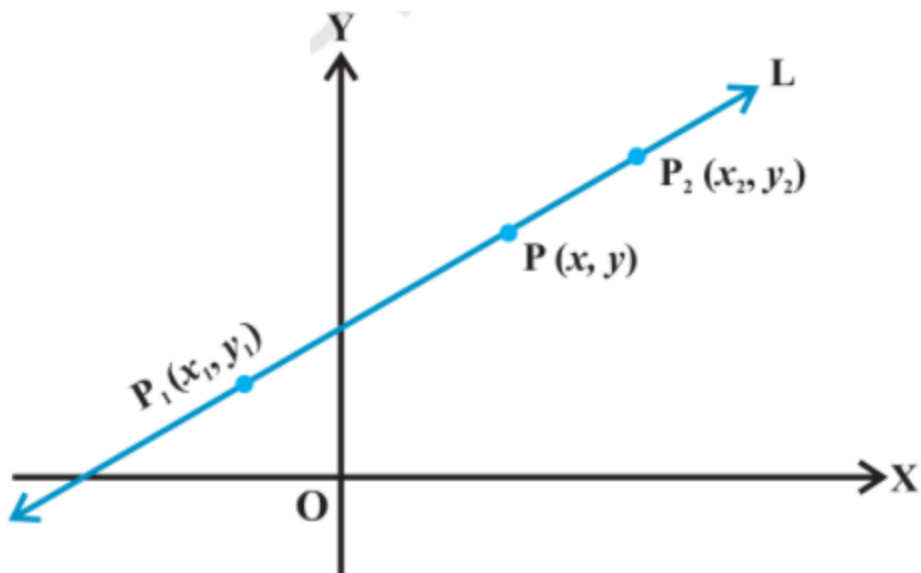
The three points P_1, P_2 and P are collinear, therefore, we have slope of $P_1P = \text{slope of } P_1P_2$

$$\text{i.e., } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

$$\text{or } y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Thus, equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$



Example Write the equation of the line through the points $(1, -1)$ and $(3, 5)$.

Solution Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5 - (-1)}{3 - 1}(x - 1)$$

or $-3x + y + 4 = 0$, which is the required equation.

Example: Find the equation of the straight line passing through the points (3,6) and (2,5).

Solution : Equation of the line is $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$$\frac{y-6}{5-6} = \frac{x-3}{-2-3}$$

$$\frac{y-6}{-1} = \frac{x-3}{-5}$$

$$5y - 30 = x - 3$$

$$x - 5y - 3 + 30 = 0$$

$$\text{Equation of the line is } x - 5y + 27 = 0$$

Intercept - form Suppose a line L makes x -intercept a and y -intercept b on the axes. Obviously L meets x -axis at the point $(a, 0)$ and y -axis at the point $(0, b)$ (Fig .10.16). By two-point form of the equation of the line, we have

$$y - 0 = \frac{b - 0}{0 - a} (x - a) \quad \text{or} \quad ay = -bx + ab$$

i.e.,
$$\frac{x}{a} + \frac{y}{b} = 1.$$

Thus, equation of the line making intercepts a and b on x - and y -axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots (5)$$

Example Find the equation of the line, which makes intercepts -3 and 2 on the x - and y -axes respectively.

Solution Here $a = -3$ and $b = 2$. By intercept form (5) above, equation of the line is

$$\frac{x}{-3} + \frac{y}{2} = 1 \quad \text{or} \quad 2x - 3y + 6 = 0.$$

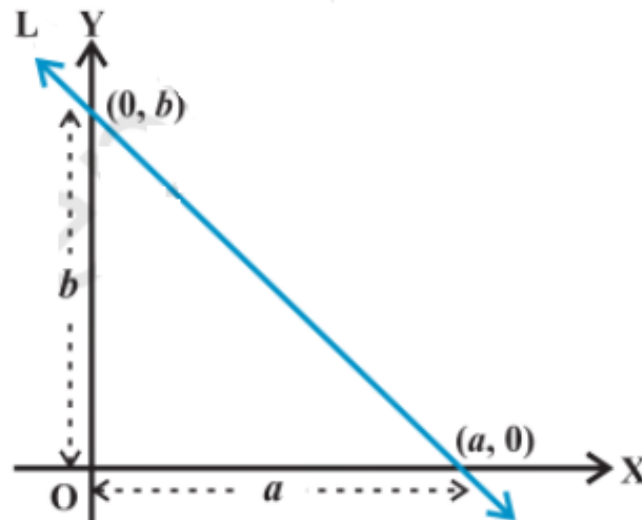


Fig 10.16

Example: Find the intercepts cut off by the line $2x - 3y + 5 = 0$ on the axes.

Solution

x – intercept: put $y = 0$

$$\therefore 2x + 5 = 0$$

$$x = \frac{-5}{2} \text{ This is the x – intercept}$$

y – intercept: Put $x = 0$

$$-3y + 5 = 0$$

$$\therefore y = \frac{5}{3} \text{ This is the y – intercept}$$

Normal form Suppose a line is known to us with following data:

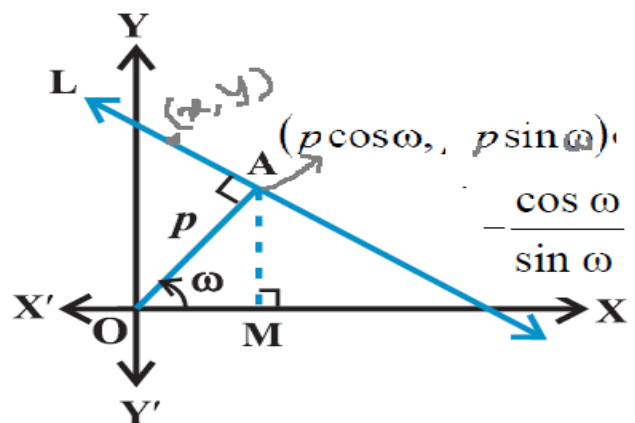
- (i) Length of the perpendicular (normal) from origin to the line.
- (ii) Angle which normal makes with the positive direction of x -axis.

Let L be the line, whose perpendicular distance from origin O be $OA = p$ and the angle between the positive x -axis and OA be $\angle XOA = \omega$.

Now, our purpose is to find slope of L and a point on it. Draw perpendicular AM on the x -axis

$$\frac{OM}{p} = \cos \omega$$

$$\frac{MA}{p} = \sin \omega,$$



we have $OM = p \cos \omega$ and $MA = p \sin \omega$, so that the coordinates of the point A are $(p \cos \omega, p \sin \omega)$.

Further, line L is perpendicular to OA . Therefore

$$\text{The slope of the line } L = -\frac{1}{\text{slope of } OA} = -\frac{1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

$$\frac{y - p \sin \omega}{(x - p \cos \omega)} = -\frac{\cos \omega}{\sin \omega} \quad \text{or} \quad x \cos \omega + y \sin \omega = p(\sin^2 \omega + \cos^2 \omega)$$

$$x \cos \omega + y \sin \omega = p.$$

Example Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x -axis is 15° .

Solution Here, we are given $p = 4$ and $\omega = 15^\circ$

$$\text{Now } \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

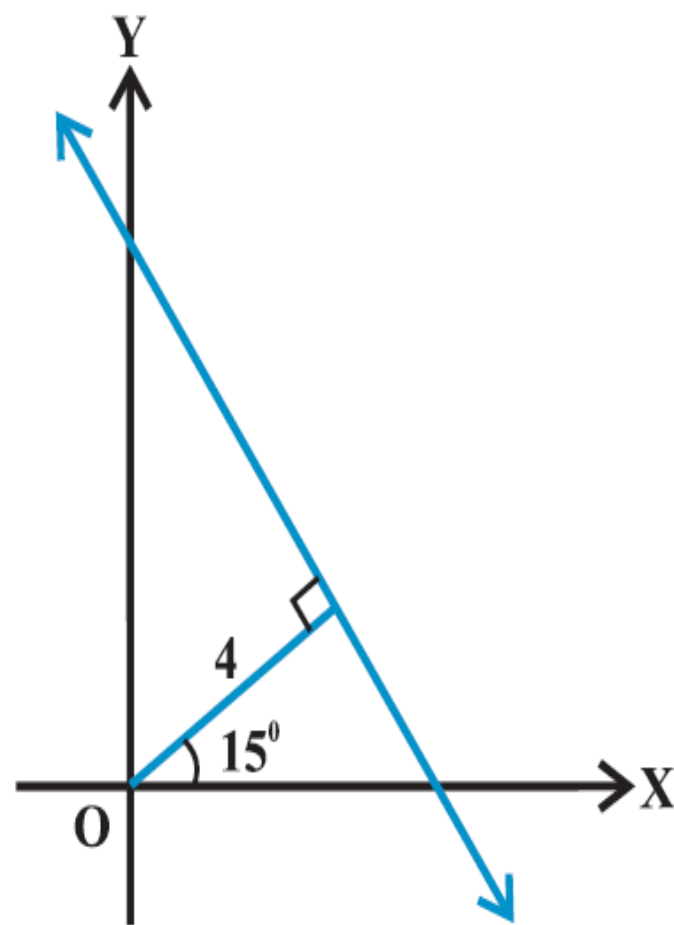
$$\text{and } \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

By the normal form, the equation of the line is

$$x \cos 15^\circ + y \sin 15^\circ$$

$$\frac{\sqrt{3}+1}{2\sqrt{2}}x + \frac{\sqrt{3}-1}{2\sqrt{2}}y = 4$$

$$(\sqrt{3}+1)x + (\sqrt{3}-1)y = 8\sqrt{2}. \text{ This is the required equation.}$$



Slope-intercept form

Case I Suppose a line L with slope m cuts the y -axis at a distance c from the origin

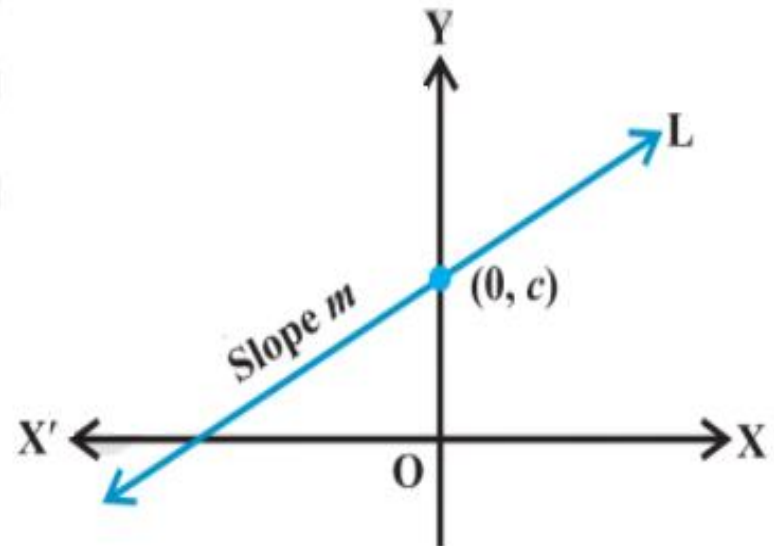
The distance c is called the *y-intercept* of the line L . Obviously, coordinates of the point where the line meet the y -axis are $(0, c)$. Thus, L has slope m and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

$$y - c = m(x - 0) \quad \text{or} \quad y = mx + c$$

Thus, the point (x, y) on the line with slope m and y -intercept c lies on the line if and only if

$$y = mx + c$$

Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y -axis, respectively.



Case II Suppose line L with slope m makes x -intercept d . Then equation of L is
 $y = m(x - d)$... (4)

Students may derive this equation themselves by the same method as in Case I.

Solution (i) Here, slope of the line is $m = \tan \theta = \frac{1}{2}$ and y - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \text{ or } 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan \theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x - 4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

General Equation of a Line

In earlier classes, we have studied general equation of first degree in two variables, $Ax + By + C = 0$, where A, B and C are real constants such that A and B are not zero simultaneously. Graph of the equation $Ax + By + C = 0$ is always a straight line.

Therefore, any equation of the form $Ax + By + C = 0$, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

10.4.1 Different forms of $Ax + By + C = 0$ The general equation of a line can be reduced into various forms of the equation of a line, by the following procedures:

(a) Slope-intercept form If $B \neq 0$, then $Ax + By + C = 0$ can be written as

$$y = -\frac{A}{B}x - \frac{C}{B} \text{ or } y = mx + c \quad \dots (1)$$

where $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.

We know that Equation (1) is the slope-intercept form of the equation of a line whose slope is $-\frac{A}{B}$, and y-intercept is $-\frac{C}{B}$.

If $B = 0$, then $x = -\frac{C}{A}$, which is a vertical line whose slope is undefined and

x-intercept is $-\frac{C}{A}$.

(b) Intercept form If $C \neq 0$, then $Ax + By + C = 0$ can be written as

$$-\frac{x}{\frac{C}{A}} + -\frac{y}{\frac{C}{B}} = 1 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 1 \quad \dots (2)$$

where $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$.

We know that equation (2) is intercept form of the equation of a line whose

x -intercept is $-\frac{C}{A}$ and y -intercept is $-\frac{C}{B}$.

If $C = 0$, then $Ax + By + C = 0$ can be written as $Ax + By = 0$, which is a line passing through the origin and, therefore, has zero intercepts on the axes.

(c) Normal form Let $x \cos \omega + y \sin \omega = p$ be the normal form of the line represented by the equation $Ax + By + C = 0$ or $Ax + By = -C$. Thus, both the equations are

same and therefore,
$$\frac{A}{\cos \omega} = \frac{B}{\sin \omega} = -\frac{C}{p}$$

which gives
$$\cos \omega = -\frac{Ap}{C} \text{ and } \sin \omega = -\frac{Bp}{C}.$$

Now
$$\sin^2 \omega + \cos^2 \omega = \left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$$

or
$$p^2 = \frac{C^2}{A^2 + B^2} \quad \text{or} \quad p = \pm \frac{C}{\sqrt{A^2 + B^2}}$$

Therefore
$$\cos \omega = \pm \frac{A}{\sqrt{A^2 + B^2}} \text{ and } \sin \omega = \pm \frac{B}{\sqrt{A^2 + B^2}}$$

Thus, the normal form of the equation $Ax + By + C = 0$ is

$$x \cos \omega + y \sin \omega = p,$$

where
$$\cos \omega = \pm \frac{A}{\sqrt{A^2 + B^2}}, \quad \sin \omega = \pm \frac{B}{\sqrt{A^2 + B^2}} \quad \text{and} \quad p = \pm \frac{C}{\sqrt{A^2 + B^2}}$$

Proper choice of signs is made so that p should be positive.

Example 13 Equation of a line is $3x - 4y + 10 = 0$. Find its (i) slope, (ii) x - and y -intercepts.

Solution (i) Given equation $3x - 4y + 10 = 0$ can be written as

$$y = \frac{3}{4}x + \frac{5}{2} \quad \dots (1)$$

Comparing (1) with $y = mx + c$, we have slope of the given line as $m = \frac{3}{4}$.

(ii) Equation $3x - 4y + 10 = 0$ can be written as

$$3x - 4y = -10 \quad \text{or} \quad \frac{x}{-\frac{10}{3}} + \frac{y}{\frac{5}{2}} = 1 \quad \dots (2)$$

Comparing (2) with $\frac{x}{a} + \frac{y}{b} = 1$, we have x -intercept as $a = -\frac{10}{3}$ and

y -intercept as $b = \frac{5}{2}$.

Example 14 Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of p and ω .

Solution Given equation is

$$\sqrt{3}x + y - 8 = 0 \quad \dots (1)$$

Dividing (1) by $\sqrt{(\sqrt{3})^2 + (1)^2} = 2$, we get

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4 \quad \text{or} \quad \cos 30^\circ x + \sin 30^\circ y = 4 \quad \dots (2)$$

Comparing (2) with $x \cos \omega + y \sin \omega = p$, we get $p = 4$ and $\omega = 30^\circ$.

Example 15 Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

Solution Given lines are

$$y - \sqrt{3}x - 5 = 0 \text{ or } y = \sqrt{3}x + 5 \quad \dots (1)$$

and $\sqrt{3}y - x + 6 = 0 \text{ or } y = \frac{1}{\sqrt{3}}x - 2\sqrt{3} \quad \dots (2)$

Slope of line (1) is $m_1 = \sqrt{3}$ and slope of line (2) is $m_2 = \frac{1}{\sqrt{3}}$.

The acute angle (say) θ between two lines is given by

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \quad \text{Putting the values of } m_1 \text{ and } m_2$$

$$\tan \theta = \left| \frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}} \right| = \left| \frac{1 - 3}{2\sqrt{3}} \right| = \frac{1}{\sqrt{3}}$$

which gives $\theta = 30^\circ$.

Example 16 Show that two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $b_1, b_2 \neq 0$ are:

(i) Parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and (ii) Perpendicular if $a_1a_2 + b_1b_2 = 0$.

Solution Given lines can be written as

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \quad \dots (1)$$

and $y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2} \quad \dots (2)$

Slopes of the lines (1) and (2) are $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$, respectively. Now

(i) Lines are parallel, if $m_1 = m_2$, which gives

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2} \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

(ii) Lines are perpendicular, if $m_1 \cdot m_2 = -1$, which gives

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = -1 \text{ or } a_1a_2 + b_1b_2 = 0$$

Example 17 Find the equation of a line perpendicular to the line $x - 2y + 3 = 0$ and passing through the point $(1, -2)$.

Solution Given line $x - 2y + 3 = 0$ can be written as

$$y = \frac{1}{2}x + \frac{3}{2} \quad \dots(1)$$

Slope of the line (1) is $m_1 = \frac{1}{2}$. Therefore, slope of the line perpendicular to line (1) is

$$m_2 = -\frac{1}{m_1} = -2$$

Equation of the line with slope -2 and passing through the point $(1, -2)$ is

$$y - (-2) = -2(x - 1) \quad \text{or} \quad y = -2x,$$

which is the required equation.

Thus, the perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Thus, the distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}.$$

If lines are given in general form, i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$,

then above formula will take the form $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$

Example 18 Find the distance of the point $(3, -5)$ from the line $3x - 4y - 26 = 0$.

Solution Given line is $3x - 4y - 26 = 0$... (1)

Comparing (1) with general equation of line $Ax + By + C = 0$, we get

$$A = 3, B = -4 \text{ and } C = -26.$$

Given point is $(x_1, y_1) = (3, -5)$. The distance of the given point from given line is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3.3 + (-4)(-5) - 26|}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}.$$

Example 19 Find the distance between the parallel lines $3x - 4y + 7 = 0$ and

$$3x - 4y + 5 = 0$$

Solution Here $A = 3$, $B = -4$, $C_1 = 7$ and $C_2 = 5$. Therefore, the required distance is

$$d = \frac{|7 - 5|}{\sqrt{3^2 + (-4)^2}} = \frac{2}{5}.$$

Example 15 The distance of the point P (1, - 3) from the line $2y - 3x = 4$ is

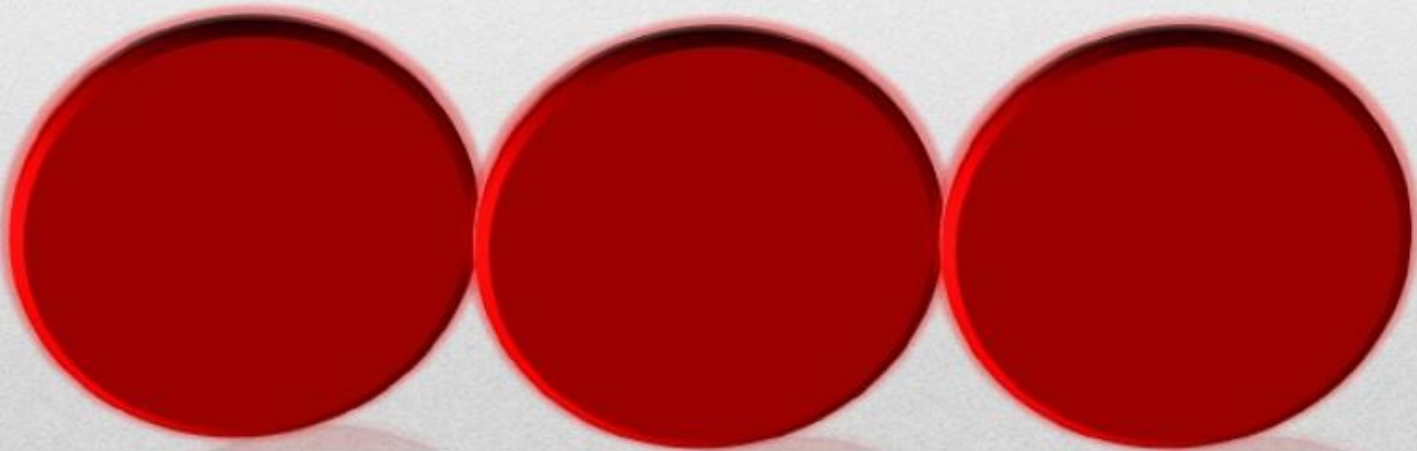
- (A) 13 (B) $\frac{7}{13}\sqrt{13}$ (C) $\sqrt{13}$ (D) None of these

Solution (A) is the correct answer. The distance of the point P (1, - 3) from the line $2y - 3x - 4 = 0$ is the length of perpendicular from the point to the line which is given by

$$\left| \frac{2(-3) - 3 - 4}{\sqrt{13}} \right| = \sqrt{13}$$

What is a circle?

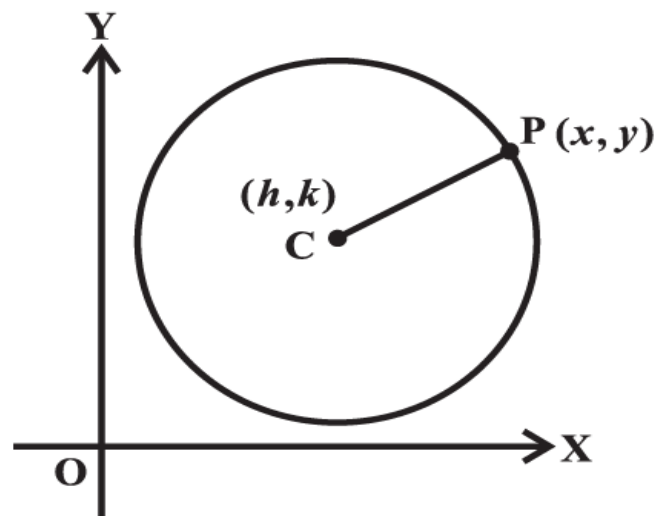
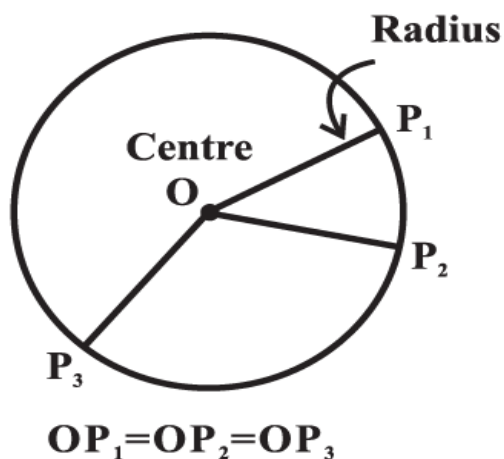
Circles are simple closed curves which divide the plane into two regions: an interior and an exterior.



Circle

A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.

The fixed point is called the *centre of the circle* and the distance from the centre to a point on the circle is called the *radius* of the circle.



The equation of the circle is simplest if the centre of the circle is at the origin.

Given $C(h, k)$ be the centre and r the radius of circle. Let $P(x, y)$ be any point on the circle (Fig11.12). Then, by the definition, $|CP| = r$. By the distance formula, we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

i.e. $(x-h)^2 + (y-k)^2 = r^2$

Equation of a Circle in General Form

The general equation of any type of circle is represented by:

$$x^2 + y^2 + 2gx + 2fy + c = 0, \text{ for all values of } g, f \text{ and } c.$$

Adding $g^2 + f^2$ on both sides of the equation gives,

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c \dots\dots\dots(1)$$

Since, $(x+g)^2 = x^2 + 2gx + g^2$ and $(y+f)^2 = y^2 + 2fy + f^2$ substituting the values in equation (1), we have

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c \dots\dots\dots(2)$$

Comparing (2) with $(x-h)^2 + (y-k)^2 = a^2$, where (h, k) is the centre and 'a' is the radius of the circle.

$$h = -g, k = -f$$

$$a^2 = g^2 + f^2 - c$$

Therefore,

$x^2 + y^2 + 2gx + 2fy + c = 0$, represents the circle with centre $(-g, -f)$ and radius equal to $a^2 = g^2 + f^2 - c$.

- If $g^2 + f^2 > c$, then the radius of the circle is real.
- If $g^2 + f^2 = c$, then the radius of the circle is zero which tells us that the circle is a point which coincides with the centre. Such type of circle is called a point circle
- $g^2 + f^2 < c$, then the radius of the circle become imaginary. Therefore, it is a circle having a real centre and imaginary radius.

Example 1 Find an equation of the circle with centre at $(0,0)$ and radius r .

Solution Here $h = k = 0$. Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

Example 2 Find the equation of the circle with centre $(-3, 2)$ and radius 4.

Solution Here $h = -3$, $k = 2$ and $r = 4$. Therefore, the equation of the required circle is

$$(x + 3)^2 + (y - 2)^2 = 16$$

Example 3 Find the centre and the radius of the circle $x^2 + y^2 + 8x + 10y - 8 = 0$

Solution The given equation is

$$(x^2 + 8x) + (y^2 + 10y) = 8$$

Now, completing the squares within the parenthesis, we get

$$(x^2 + 8x + 16) + (y^2 + 10y + 25) = 8 + 16 + 25$$

i.e.
$$(x + 4)^2 + (y + 5)^2 = 49$$

i.e.
$$\{x - (-4)\}^2 + \{y - (-5)\}^2 = 7^2$$

Therefore, the given circle has centre at $(-4, -5)$ and radius 7.

Example 4 Find the equation of the circle which passes through the points $(2, -2)$, and $(3, 4)$ and whose centre lies on the line $x + y = 2$.

Solution Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Since the circle passes through $(2, -2)$ and $(3, 4)$, we have

$$(2 - h)^2 + (-2 - k)^2 = r^2 \quad \dots (1)$$

and $(3 - h)^2 + (4 - k)^2 = r^2 \quad \dots (2)$

Also since the centre lies on the line $x + y = 2$, we have

$$h + k = 2 \quad \dots (3)$$

Solving the equations (1), (2) and (3), we get

$$h = 0.7, \quad k = 1.3 \quad \text{and} \quad r^2 = 12.58$$

Hence, the equation of the required circle is

$$(x - 0.7)^2 + (y - 1.3)^2 = 12.58.$$

Example

Equation of a circle is $x^2+y^2-12x-16y+19=0$. Find the centre and radius of the circle.

Solution:

Given equation is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$,

$$2g = -12, 2f = -16, c = 19$$

$$g = -6, f = -8$$

Centre of the circle is (6,8)

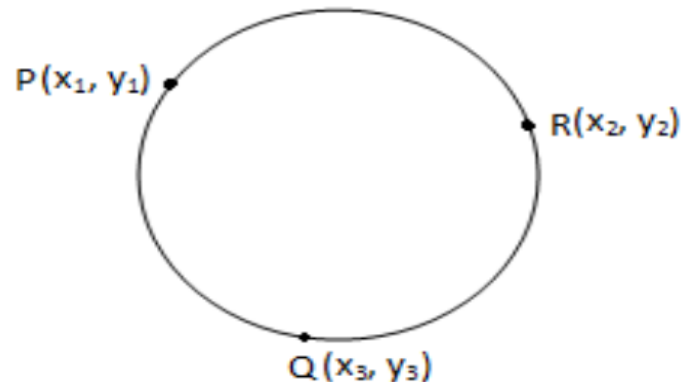
$$\text{Radius of the circle} = \sqrt{[(-6)^2 + (-8)^2 - 19]} = \sqrt{[100 - 19]} =$$

$$= \sqrt{81} = 9 \text{ units.}$$

Therefore, the radius of the circle is 9 units.

Circle Passing Through Three Given Points

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ are the three given points.



Let the equation of the general form of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

According to the problem, the above equation of the circle passes through the points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$. Therefore,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \text{ (ii)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \text{ (iii)}$$

$$\text{and } x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \text{ (iv)}$$

Form the above three equations (ii), (iii) and (iv) find the value of g , f and c . Then substituting the values of g , f and c in (i) we can find the required equation of the circle.

2. Find the equation of the circle passes through three points (1, - 6), (2, 1) and (5, 2). Also find the co-ordinate of its centre and the length of the radius.

Solution: Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots(i)$$

According to the problem, the above equation passes through the coordinate points (1, - 6), (2, 1) and (5, 2).

Therefore, substituting the coordinates of three points (1, - 6), (2, 1) and (5, 2) successively in equation (i) we get,

$$\text{For the point (1, - 6): } 1 + 36 + 2g - 12f + c = 0$$

$$\Rightarrow 2g - 12f + c = -37 \dots\dots\dots(ii)$$

$$\text{For the point (2, 1): } 4 + 1 + 4g + 2f + c = 0$$

$$\Rightarrow 4g + 2f + c = -5 \dots\dots\dots(iii)$$

$$\text{For the point (5, 2): } 25 + 4 + 10g + 4f + c = 0$$

$$\Rightarrow 10g + 4f + c = -29 \dots\dots\dots(iv)$$

Subtracting (ii) from (iii) we get,

$$2g + 14f = 32$$

$$\Rightarrow g + 7f = 16 \text{(v)}$$

Again, Subtracting (ii) from (iv) we get,

$$8g + 16f = 8$$

$$\Rightarrow g + 2f = 1 \text{(vi)}$$

Now, solving equations (v) and (vi) we get, $g = -5$ and $f = 3$.

Putting the values of g and f in (iii) we get, $c = 9$.

Therefore, the equation of the required circle is $x^2 + y^2 - 10x + 6y + 9 = 0$

Thus, the co-ordinates of its centre are $(-g, -f) = (5, -3)$ and radius =
 $\sqrt{g^2 + f^2 - c} = \sqrt{25 + 9 - 9}$
 $= \sqrt{25} = 5$ units.

Equation of a Circle when the Line Segment Joining Two Given Points is a Diameter

the equation of the circle drawn on the straight line joining two given points (x_1, y_1) and (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Find the equation of circle whose diameter is the line joining the points $(-4, 3)$ and $(12, -1)$.

The required circle by rule

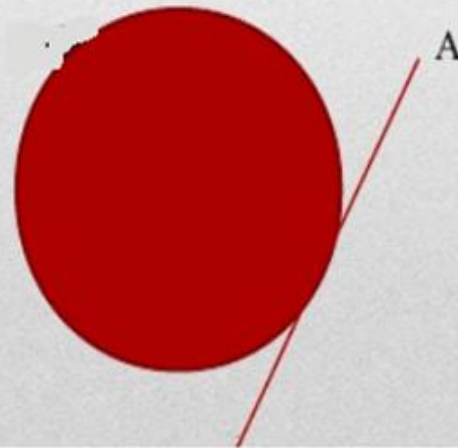
$$(x + 4)(x - 12) + (y - 3)(y + 1) = 0$$

$$\text{or } x^2 + y^2 - 8x - 2y - 51 = 0$$

$$\therefore g = -4, f = -1, c = -51$$

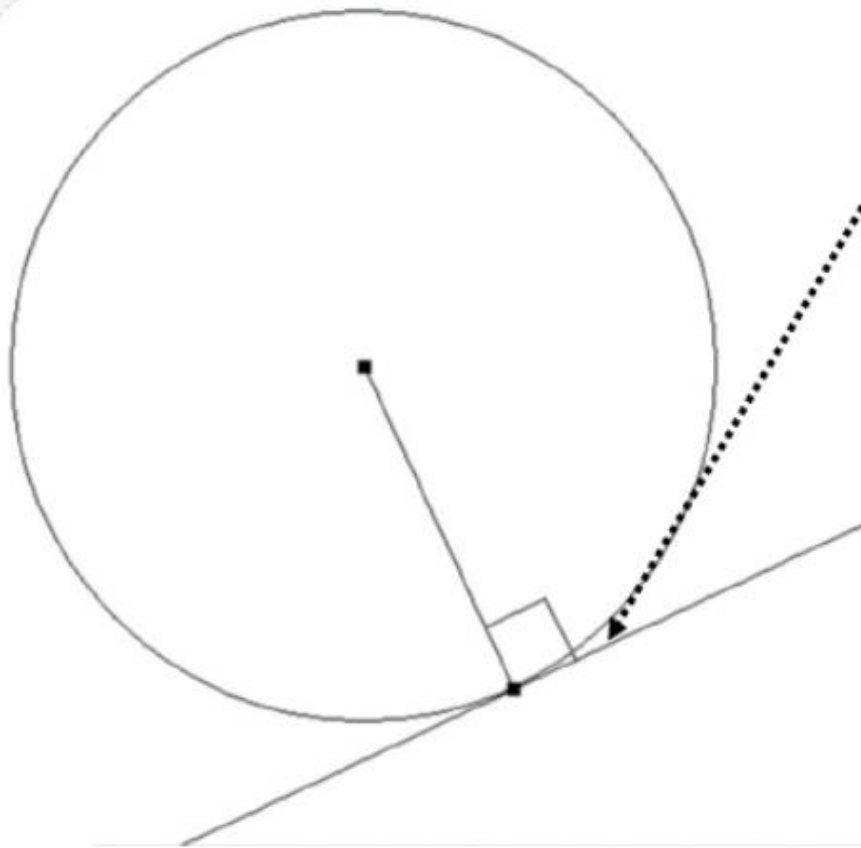
TANGENT

**A TANGENT TO A CIRCLE IS A
LINE THAT INTERSECTS THE
CIRCLE AT ONLY ONE
point**



A circle has infinitely many tangents, touching the circle at infinite points on its circumference.

Tangent & Radius

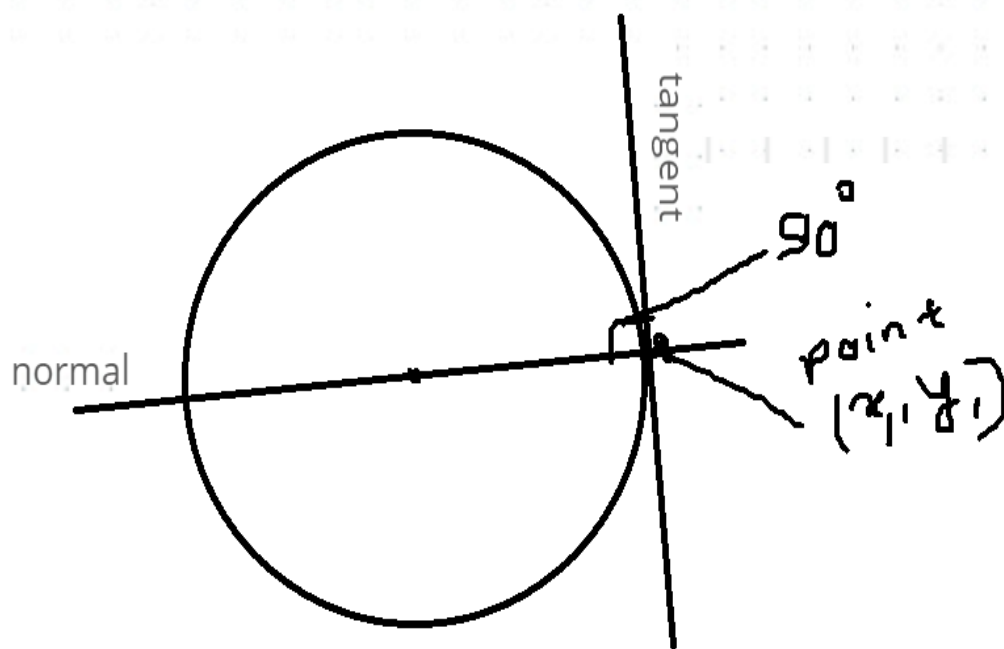


A tangent is perpendicular to the radius of a circle

Normal: Definition

The normal to a curve at a given point is the line perpendicular to the tangent at that point.

In other words, the line perpendicular to the tangent (to a curve), and passing through the point of contact, is known as the normal.



How to Find Tangent and Normal to a Circle

In geometry, tangents and normals have great importance. A straight line that touches a circle at only one point is the tangent of the circle. The normal to a circle is a straight line drawn at 90° to the tangent at the point where the tangent touches the circle. A normal will pass through the centre of the circle. Tangents and normals to a circle is an important topic. In this article, we will discuss how to find the tangent and normal to a circle.

Steps to find Tangent and Normal to a Circle

Step 1. If $x^2 + y^2 = a^2$ is a circle, then

- The equation of a tangent to the circle at (x_1, y_1) is given by $xx_1 + yy_1 = a^2$.
- The equation of normal to the circle at (x_1, y_1) is given by $yx_1 - xy_1 = 0$.
- The equation of a tangent to the circle at $(a \cos \theta, a \sin \theta)$ is given by $x \cos \theta + y \sin \theta = a$.
- The equation of a normal to the circle at $(a \cos \theta, a \sin \theta)$ is given by $x \sin \theta - y \cos \theta = 0$.

Step 2. If the circle is given by $x^2 + y^2 + 2gx + 2fy + c = 0$

- The equation of a tangent to the circle at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- The equation of normal to the circle at (x_1, y_1) is $(y - y_1)/(y_1 + f) = (x - x_1)/(x_1 + g)$.

Step 3. For a line $y = mx + c$ to be a tangent to a circle $x^2 + y^2 = a^2$, it should satisfy $c = \pm a(\sqrt{1+m^2})$. The equation of tangent is given by $y = mx \pm a(\sqrt{1+m^2})$

